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Article

On S-2-Prime Ideals of Commutative Rings

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Abstract: Prime ideals and their generalizations are crucial in numerous research areas, particularly in commutative algebra. The concept of generalization of prime ideals begins with the study of weakly prime ideals. Since then, subsequent works aimed at expanding this concept into more generalized forms. Among these, S-prime ideals and 2-prime ideals have reaped attention recently. This paper aims to characterize S-2-prime ideals, which serve as a generalization encompassing both 2-prime ideals and S-prime ideals. To accomplish this objective, we construct an ideal which distinct from a multiplicatively closed subset with the help of commutative rings. We investigate the localization and the S-2-prime avoidance lemma in commutative rings. Furthermore, we explore the properties of this class of ideals in trivial ring extensions and amalgamated algebras along an ideal. We delve into S-properties for compactly packedness, compactly 2-packedness and coprimely packedness in trivial ring extentions. Moreover, this notion of ideals helps us to indicate that many results stated in S-prime ideals and 2-prime ideals can be readily expanded to the framework of S-2-prime ideals. Supporting examples also highlight a significant distinction between S-2-prime ideals and stated ideals.

Keywords: S-prime ideals; 2-prime ideals; S-2-prime ideals

MSC: 13A15; 13C05; 13A99

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1. Introduction

In this paper, we consider a commutative ring with identity denoted by R. Prime ideals play a crucial role in commutative rings and take part in applications in various areas such as graph theory, cryptology and algebraic geometry, etc. Several characterizations of prime ideals exist in the literature such as Dedekind's characterization, and there have been numerous studies generalizing the concept of prime ideals. Among these, in [1] Anderson and Smith defined weakly prime ideals provided that $0 \neq \alpha\beta \in Q$ for some α, $\beta \in R$ implies $\alpha \in Q$ or $\beta \in Q$. Later, Hamed and Malek [2] introduced S-prime ideals and studied them over S-Noetherian rings. Recall that a subset S of R is called a multiplicatively closed subset (in briefly m.c.s) if S is closed under multiplication with $1 \in S$ and $0 \notin S$. Let S be an m.c.s of R and Q be a proper ideal with $Q \cap S = \emptyset$. Then, Q is called an S-prime ideal of R if $\alpha\beta \in Q$ for some $\alpha, \beta \in R$, then there exists $s \in S$ such that $s\alpha \in Q$ or $s\beta \in Q$. Subsequently, Almahdi et al. [3] (and then in [4] Massaoud) presented weakly S-prime ideals (resp. S-primary ideals) defining them as ideals Q such that if $0 \neq \alpha\beta \in Q$ (resp. $\alpha\beta \in Q$) for $\alpha, \beta \in R$, then either $s\alpha \in Q$ or $s\beta \in Q$ (resp. $s\alpha \in Q$ or $s\beta \in \sqrt{Q}$). Then, in [5], Beddani and Messirdi and, in [6], Nikhandish et al. (and then in [7] Koç) described the concept of 2-prime ideals (resp. weakly 2-prime ideals) which is a different generalization of prime ideals and studied this class of ideals over valuation rings. Q is called a 2-prime (resp. weakly 2-prime) ideal of R if $\alpha\beta \in Q$ (resp. $0 \neq \alpha\beta \in Q$) for $\alpha, \beta \in R$, then either $\alpha^2 \in Q$ or $\beta^2 \in Q$.

Moreover, strongly quasi-primary ideals, being an intermediate class of primary ideals and quasi-primary ideals, [8] were investigated and examined in terms of graphs. A proper ideal Q is called a strongly quasi-primary ideal if $\alpha\beta \in Q$ for $\alpha, \beta \in R$ implies either $\alpha^2 \in Q$ or $\beta^n \in Q$ ($\alpha^n \in Q$ or $\beta^2 \in Q$) for some $n \in \mathbb{N}$. Furthermore, $(\delta, 2)$-primary ideals which are expansion of 2-prime ideals were presented by [9]. We call a proper ideal Q of R a $(\delta, 2)$-primary ideal if $\alpha, \beta \in R$ and $\alpha\beta \in Q$, then $\alpha^2 \in Q$ or $\beta^2 \in \delta(Q)$. Another generalization of prime ideals was found in 1-absorbing prime ideals, which were characterized in the context of Noetherian divided rings and von Neumann regular rings [10]. A proper ideal Q of R is said to be a 1-absorbing prime ideal if $\alpha\beta\gamma \in Q$ for some non-units $\alpha, \beta, \gamma \in R$, then $\alpha\beta \in Q$ or $\gamma \in Q$. Moreover, in [11], the notion of S-semiprime ideals which are the generalization of semiprime ideals and another generalization of prime ideals was investigated in the scope of the relations between S-semiprime ideals and prime, semiprime, maximal ideals. An ideal Q of R is said to be an S-semiprime ideal if there exists $s \in S$ and whenever $\alpha^n \in Q$ for some $n \in \mathbb{N}$ and $\alpha \in R$, then $s\alpha \in Q$ where S is an m.c.s of R.

As for the idealization or trivial ring extension, this notion was presented in [12] (and then in [13]). Idealization or trivial ring extension is currently being studied for its applications for different algebraic structures and many new studies are being carried out. Additionally, amalgamation duplication and its generalization, the amalgamation along an ideal concerning a ring homomorphism, were studied and introduced by [14,15]. These constructions provide insights into different types of ideals from a comprehensive perspective. On the other hand, the notion of the union of prime ideals was investigated by [16,17] as well as [18]. Then, these notions and the union of 2-prime ideals were explored for trivial ring extensions by [7]. Another structure on which extensive studies have been carried out to date is commuting maps. In the latest study [19], commuting maps are explored on alternative division rings which have no characteristic two.

Since generalizations of prime ideals are a rapidly continuing field of study today, we want to make a broader generalization with the help of 2-prime ideals and S-prime ideals. The above mentioned algebraic structures, the study in [19], and applications of the work on [8] to graphs motivated and inspired us. To achieve this purpose, we introduce and examine the concept of S-2-prime ideals in commutative rings in the section after the introduction. We call a proper ideal Q of R with $Q \cap S = \emptyset$ an S-2-prime ideal of R if $\alpha\beta \in Q$ for some $\alpha, \beta \in R$, then there exists $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. Then, we explore the relationship among S-2-prime ideals and other classes of ideals such as S-prime, 2-prime and S-semiprime ideals. In fact, every 2-prime and S-prime ideal is an S-2-prime ideal of R but the converses do not hold (see Examples 1–3). Subsequently, we give a new characterization of S-2-prime ideals with the help of [20]. That is, we prove that Q is an S-2-prime ideal of R if and only if for every J, K two ideals of R there exists $s \in S$ provided that $JK \subseteq Q$, then $sJ_{[2]} \subseteq Q$ or $sK_{[2]} \subseteq Q$ (Theorem 2). We also investigate the properties of this ideal in terms of homomorphism, direct product ring and localization. Moreover, we establish a theorem analogous to the celebrated prime avoidance lemma (Theorem 4). The S_p-2-prime ideals of the ring $\mathbb{Z}_n$ are totally characterized (Theorem 5). Additionally, we delve into S-2-prime ideals in trivial ring extensions and amalgamated algebra in the subsequent section. Section 4 is devoted to S-properties of compactly packedness, compactly 2-packedness and coprimely packedness for trivial ring extension, with support from [21,22]. Finally, the last section shows the conclusions drawn from the results presented throughout this work and recommends future research in the realm of S-2-prime ideals.

Consequently, we find that the majority of the outcomes established by S-prime ideals and 2-prime ideals are similarly achieved by S-2-prime ideals having a broader scope. Moreover, our analysis leads us to the conclusion that in trivial ring extensions, properties such as compactly packedness, compactly 2-packedness and coprimely packedness, along with their S-properties, yield equivalent results.

2. Properties of S-2-Prime Ideals

Definition 1. *Let S be an m.c.s of a ring R and Q be a proper ideal of R with $Q \cap S = \emptyset$. Then, Q is called an S-2-prime ideal of R associated with s if $\alpha\beta \in Q$ for some α, $\beta \in R$, then there exists $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$.*

Every 2-prime ideal of R disjoint with S is a S-2-prime ideal of R where S is an m.c.s of R. Provided that S consists of units of R, then 2-prime ideals and S-2-prime ideals coincide. Otherwise, these are distinct concepts.

Example 1. *Let $R = \mathbb{Z}[X]$ and $S = \{p^n : n \in \mathbb{N}\}$, where p is a prime integer. Consider $Q = pX\mathbb{Z}[X]$. It is clear that $Q \cap S = \emptyset$. Then, Q is an S-2-prime ideal of R. Indeed, if $gh \in Q = pX\mathbb{Z}[X]$ for some $g, h \in R$, then since $Q \subset X\mathbb{Z}[X]$ and $X\mathbb{Z}[X]$ is a prime ideal of R, we have $X|g$ or $X|h$. Hence, $pg^2 \in Q$ or $ph^2 \in Q$ for $s := p$. On the other hand, Q is not a 2-prime ideal as $p.X \in Q$, but $p^2 \notin Q$ and $X^2 \notin Q$.*

Example 2. *Let $R = \mathbb{Z}$, $S = \{5^n : n \in \mathbb{N}\}$ and $Q = 40\mathbb{Z}$. It is clear that $Q \cap S = \emptyset$. Then, Q is an S-2-prime ideal of R. Let $ab \in Q \subset 8\mathbb{Z}$ for some $a, b \in \mathbb{Z}$ and $s := 5$. Since $8\mathbb{Z}$ is a primary ideal, we have $a \in 8\mathbb{Z}$ or $b \in 2\mathbb{Z}$. In the former case, we conclude $sa^2 \in Q$. Assume that the latter case holds. If $b \in 4\mathbb{Z}$, then $sb^2 \in Q$. Assume $b \in 2\mathbb{Z} \backslash 4\mathbb{Z}$. Then, clearly we have $a \in 4\mathbb{Z}$ and $sa^2 \in Q$. However, Q is not a 2-prime ideal of R as $4.10 \in 40\mathbb{Z}$, but $4^2 \notin 40\mathbb{Z}$ and $10^2 \notin 40\mathbb{Z}$.*

It is clear that any S-prime ideal is an S-2-prime ideal of R. Recall from [11] that a proper ideal Q of R is said to be S-semiprime ideal of R if $a^m \in Q$ for some $a \in R$ and $m \in \mathbb{N}$, then there exists $s \in S$ such that $sa \in Q$. Note that if an S-2-prime ideal of a ring is S-semiprime ideal, then it is S-prime. However, the converse of this implication does not hold in general.

Example 3. *Consider $R = \mathbb{Z}_{18}$, $S = \{\overline{1}, \overline{5}, \overline{7}, \overline{11}, \overline{13}, \overline{17}\}$ and $Q = \{\overline{0}, \overline{9}\}$. If $\alpha\beta \in Q \subset \overline{3}\mathbb{Z}_{18}$ for some α, $\beta \in R$, then $\alpha \in \overline{3}\mathbb{Z}_{18}$ or $b \in \overline{3}\mathbb{Z}_{18}$. Then for $s := 1$, $s\alpha^2 \in Q$ or $s\beta^2 \in Q$ and Q is an S-2-prime ideal of R. On the other hand, since $\overline{3}.\overline{3} \in Q$ but $s.\overline{3} \notin Q$ for each $s \in S$, Q is not an S-prime ideal of R.*

Let S be an m.c.s of R and Q be an ideal of R with $Q \cap S = \emptyset$. Suppose that $s \in S$ and $\overline{S} = \{\overline{s} : s \in S\}$ where $\overline{s}$ refers the equivalence class of s in R/Q. Clearly, $\overline{S}$ is an m.c.s of R/Q. It is not difficult to see that if $Z(R/Q) \cap \overline{S} = \emptyset$ where $Z(R/Q)$ is the set of zero divisors of R/Q, then 2-prime ideals and S-2-prime ideals are coincide. In this case, we have $(Q : s) = Q$ for each $s \in S$.

Theorem 1. *Let S be an m.c.s and Q be a proper ideal of R with $Q \cap S = \emptyset$ and $(S^{-1}Q) \cap R \subseteq (Q : s)$. The following statements are equivalent.*

1. *Q is an S-2-prime ideal of R.*
2. *$(Q : s)$ is an S-2-prime ideal of R for some $s \in S$.*
3. *$S^{-1}Q$ is a 2-prime ideal of $S^{-1}R$ and $(S^{-1}Q) \cap R = (Q : s)$ for some $s \in S$.*

Proof. 1. $\Longleftrightarrow$ 2. Suppose that Q is an S-2-prime ideal of R associated with $s' \in S$ and $\alpha\beta \in (Q : s)$ for some $\alpha, \beta \in R$. Then, $(s\alpha)\beta \in Q$ which implies that $s's^2\alpha^2 \in Q$ or $s'\beta^2 \in Q$. Put $t = ss' \in S$. Then, $t\alpha^2 \in (Q : s)$ or $t\beta^2 \in Q \subseteq (Q : s)$, and so $(Q : s)$ is an S-2-prime ideal of R associated with t. Conversely, suppose that $(Q : s)$ is an S-2-prime ideal associated with $s' \in S$ and $\alpha, \beta \in R$ with $\alpha\beta \in Q$. Then, $s'\alpha^2 \in (Q : s)$ or $s'\beta^2 \in (Q : s)$. Thus, Q is an S-2-prime ideal of R associated with $s'' = s's$.

1. $\Longleftrightarrow$ 3. Let $\frac{\alpha}{\widetilde{s_1}}$, $\frac{\beta}{\widetilde{s_2}} \in S^{-1}R$ with $\frac{\alpha}{\widetilde{s_1}}\frac{\beta}{\widetilde{s_2}} \in S^{-1}Q$. This implies $s'\alpha\beta \in Q$ for some $s' \in S$. Then, there exists $s \in S$ such that $ss'^2\alpha^2 \in Q$ or $s\beta^2 \in Q$. It refers that $(\frac{\alpha}{\widetilde{s_1}})^2 = \frac{ss'^2\alpha^2}{ss'^2\widetilde{s_1}^2} \in S^{-1}Q$ or $(\frac{\beta}{\widetilde{s_2}})^2 = \frac{s\beta^2}{s\widetilde{s_2}^2} \in S^{-1}Q$ and $S^{-1}Q$ is a 2-prime ideal of $S^{-1}R$. Conversely, let α, $\beta \in R$

with $\alpha\beta \in Q$. Then, $\frac{\alpha}{1}\frac{\beta}{1} \in S^{-1}Q$, and we have $(\frac{\alpha}{1})^2 \in S^{-1}Q$ or $(\frac{\beta}{1})^2 \in S^{-1}Q$. Then, $s'\alpha^2 \in Q$ for some $s' \in S$ or $s''\beta^2 \in Q$ for some $s'' \in S$. In the former case, we have $\alpha^2 = \frac{s'\alpha^2}{s'} \in \left(S^{-1}Q\right) \cap R \subseteq (Q : s)$ because of our assumption. Therefore, $s\alpha^2 \in Q$, and in the latter case, we conclude similarly that $s\beta^2 \in Q$. $\square$

Recall that the ideal generated by n^{th} powers of elements of a proper ideal Q is denoted by $Q_{[n]} = \langle \{a^n : a \in Q\} \rangle$. It can be seen that $Q_{[n]} \subseteq Q^n \subseteq Q$ and also the equality provides if $n = 1$. Provided that $n!.1_R$ is a unit of R, then $Q_{[n]} = Q^n$ Definition 1 and Theorem 5 [20].

Theorem 2. *Let S be an m.c.s of R and Q be a proper ideal of R with $Q \cap S = \emptyset$. Q is an S-2-prime ideal of R if and only if for every J, K two ideals of R there exists $s \in S$ provided that $JK \subseteq Q$, then $sJ_{[2]} \subseteq Q$ or $sK_{[2]} \subseteq Q$. In particular, if $2!.1_R$ is a unit of R, then Q is an S-2-prime ideal of R if and only if for every J, K two ideals of R there exists $s \in S$ provided that $JK \subseteq Q$, then $sJ^2 \subseteq Q$ or $sK^2 \subseteq Q$.*

Proof. Assume on the contrary that for each $m \in S$, there exist J_m, K_m where they are ideals of R such that $J_mK_m \subseteq Q$ but $mJ_{m[2]} \nsubseteq Q$ and $mK_{m[2]} \nsubseteq Q$. Then, there is $\alpha_s \in J_s$ and $\beta_s \in K_s$ with $s\alpha_s^2 \notin Q$ and $s\beta_s^2 \notin Q$. On the other hand, $\alpha_s\beta_s \in J_sK_s \subseteq Q$, and we have a contradiction. Conversely, suppose that $\alpha\beta \in Q$ for some α, $\beta \in R$. We have, $(\alpha R)(\beta R) \subseteq Q$, and the assumption yields $s \in S$ such that $s(\alpha R)_{[2]} \subseteq Q$ or $s(\beta R)_{[2]} \subseteq Q$. Thus, $s\alpha^2 \in Q$ or $s\beta^2 \in Q$, and the proof is complete. $\square$

Note that the intersection of S-2-prime ideals is not an S-2-prime ideal. For instance, consider $R = \mathbb{Z}$ with $P = p\mathbb{Z}$ and $Q = q\mathbb{Z}$, where p and q are distinct prime integers. Let $S = \{s^n : s \text{ is a prime number distinct from } p \text{ and } q, n \in \mathbb{N}\}$. Clearly, P and Q are S-2-prime ideals of R, but $P \cap Q$ is not an S-2-prime ideal of R as $p.q \in P \cap Q$ but $s.p^2 \notin P \cap Q$ and $s.q^2 \notin P \cap Q$ for all $s \in S$. The question occurs when the intersection of S-2-prime ideals will be an S-2-prime ideal. The answer to this question is that if we take S as a strongly m.c.s of R and a chain of S-2-prime ideals of R, then we can achieve the intersection of S-2-prime ideals from the chain that is an S-2-prime ideal of R.

The next proposition is given without proof as it is straightforward.

Proposition 1. *Let S be an m.c.s of R and Q, J be ideals of R with $Q \cap S = \emptyset$.*

1. *If Q is S-2-prime ideal of R, then WQ is S-2-prime ideal of R for all ideals W of R with $W \cap S \neq \emptyset$.*
2. *Let $R \subseteq \widetilde{R}$ be an extension of commutative rings. If $\widetilde{Q}$ is S-2-prime ideal of $\widetilde{R}$, then $\widetilde{Q} \cap R$ is S-2-prime ideal of R.*

Lemma 1. *Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$ where S_1 and S_2 are m.c.ss of R_1 and R_2, respectively. Let Q_1 and Q_2 are proper ideals of R_1 and R_2, respectively. Then, $Q = Q_1 \times Q_2$ is an S-2-prime ideal of R if and only if Q_1 is an S_1-2-prime ideal of R_1 with $S_2 \cap Q_2 \neq \emptyset$ or Q_2 is an S_2-2-prime ideal of R_2 with $S_1 \cap Q_1 \neq \emptyset$.*

Proof. Suppose that $Q = Q_1 \times Q_2$ is an S-2-prime ideal of R. Since $(1, 0)(0, 1) \in Q$, there exists $(s_1, s_2) \in S$ such that $(s_1, s_2)(1, 0)^2 \in Q$ or $(s_1, s_2)(0, 1)^2 \in Q$. Hence, we have either $S_1 \cap Q_1 \neq \emptyset$ or $S_2 \cap Q_2 \neq \emptyset$. We may assume $S_2 \cap Q_2 \neq \emptyset$. As $Q \cap S = \emptyset$, we have $S_1 \cap Q_1 = \emptyset$. Let $\alpha\beta \in Q_1$ for some α, $\beta \in R_1$. Choose $q \in S_2 \cap Q_2$ such that $(\alpha, q)(\beta, 1) \in Q$. Because of the assumption, there exists $(s_1, s_2) \in S$ such that $(s_1, s_2)(\alpha, q)^2 \in Q$ or $(s_1, s_2)(\beta, 1)^2 \in Q$. Hence, $s_1\alpha^2 \in Q_1$ or $s_1\beta^2 \in Q_1$ as required. Conversely, suppose that $S_1 \cap Q_1 \neq \emptyset$ and Q_2 is an S_2-2-prime ideal of R_2. Choose $s_1 \in S_1 \cap Q_1$. Let $(k, l)(m, w) = (km, lw) \in Q$ for some k, $m \in R_1$ and l, $w \in R_2$. This refers to $lw \in Q_2$, and there exists $s_2 \in S_2$ such that $s_2l^2 \in Q_2$ or $s_2w^2 \in Q_2$. Put $s = (s_1, s_2) \in S$. Then, we have $(s_1, s_2)(k, l)^2 = (s_1k^2, s_2l^2) \in Q$ or $(s_1, s_2)(m, w)^2 = (s_1m^2, s_2w^2) \in Q$. Thus, Q is an

S-2-prime ideal of R. Similarly, if $S_2 \cap Q_2 \neq \emptyset$ and Q_1 is an S_1-2-prime ideal of R_1, then we can achieve a similar conclusion. □

Theorem 3. *Let $n \geq 2$ and $R = R_1 \times \cdots \times R_n$ and $S = S_1 \times \cdots \times S_n$ where S_i is an m.c.s of R_i for $i = 1, \ldots, n$, respectively. Let Q_i is a proper ideal of R_i for $i = 1, \ldots, n$, respectively. Then, $Q = Q_1 \times \cdots \times Q_n$ is an S-2-prime ideal of R if and only if Q_m is an S_m-2-prime ideal of R_m for some $m \in \{1, \ldots, n\}$ and $Q_i \cap S_i \neq \emptyset$ for each $i \in \{1, \ldots, n\} - \{m\}$.*

Proof. We will use induction on n. For $n = 2$, the claim is true by Lemma 1. Suppose that the claim is correct for $k < n$. Let $k = n$ and $\acute{R} = R_1 \times \cdots \times R_{n-1}$, $\acute{S} = S_1 \times \cdots \times S_{n-1}$ and $\acute{Q} = Q_1 \times \cdots \times Q_{n-1}$. Since $Q = \acute{Q} \times Q_n$ with $R = \acute{R} \times R_n$ and $S = \acute{S} \times S_n$, from the assumption, we have either that Q_n is an S_n-2-prime of R_n and $\acute{Q} \cap \acute{S} \neq \emptyset$ or $\acute{Q}$ is an $\acute{S}$-2-prime ideal of $\acute{R}$ and $Q_n \cap S_n \neq \emptyset$. If $\acute{Q} \cap \acute{S} \neq \emptyset$ and Q_n is an S_n-2-prime of R_n, then the proof is complete. So, suppose that $Q_n \cap S_n \neq \emptyset$ and $\acute{Q}$ is an $\acute{S}$-2-prime ideal of $\acute{R}$. From induction hypothesis for $k = n - 1$, we have that Q_j is an S_j-2-prime ideal of R_j for some $j \in \{1, 2, \ldots, n-1\}$ and $Q_i \cap S_i \neq \emptyset$ for each $i \in \{1, 2, \ldots, n-1\} - \{j\}$. Hence, the desired condition is provided. For the converse part, suppose that Q_1 is an S_1-2-prime ideal of R_1 associated with $s' \in S_1$ and $s_i \in Q_i \cap S_i$ for each $i \neq 1$. Taking $s = (s', s_2, \ldots, s_n)$, we can easily see that $Q = Q_1 \times \cdots \times Q_n$ is an S-2-prime ideal of R. □

Let Q is an ideal of a ring R. By $Z_Q(R)$, we denote $\{z \in R : zt \in Q \text{ for some } t \notin Q\}$ [7].

Proposition 2. *Let $S, \acute{S} \subseteq R$ be m.c.ss of R and Q be a proper ideal of R. If Q is an $\acute{S}$-2-prime ideal of R with $Q \cap S = \emptyset$, then $S^{-1}Q$ is an $S^{-1}\acute{S}$-2-prime ideal of $S^{-1}R$. The converse also holds if $Z_Q(R) \cap S = \emptyset$.*

Proof. It is clear that if $\acute{S}$ is an m.c.s of R, then $S^{-1}\acute{S}$ is an m.c.s of $S^{-1}R$. Moreover, $S^{-1}Q$ is a proper ideal of $S^{-1}R$ since $Q \cap S = \emptyset$ as well as $S^{-1}Q \cap S^{-1}\acute{S} = \emptyset$. Suppose that $\frac{m}{s}\frac{n}{t} = \frac{mn}{st} \in S^{-1}Q$ for some $m, n \in R$ and $s, t \in S$. Then, $\tilde{s}(mn) = (\tilde{s}m)n \in Q$ for some $\tilde{s} \in S$. From assumption, there exists $\acute{s} \in \acute{S}$ such that $\acute{s}(\tilde{s}m)^2 \in Q$ or $\acute{s}n^2 \in Q$. This refers to the fact that $\frac{\acute{s}}{1}(\frac{m}{s})^2 = \frac{\acute{s}}{1}\frac{(\tilde{s}m)^2}{(\tilde{s}s)^2} \in S^{-1}Q$ or $\frac{\acute{s}}{1}(\frac{n}{t})^2 \in S^{-1}Q$. Conversely, it is clear that $Q \cap \acute{S} = \emptyset$. Let $mn \in Q$ for some $m, n \in R$. Then, $\frac{m}{1}\frac{n}{1} \in S^{-1}Q$, and from the assumption, there exists $\frac{\acute{s}}{s} \in S^{-1}\acute{S}$ such that $\frac{\acute{s}}{s}(\frac{m}{1})^2 \in S^{-1}Q$ or $\frac{\acute{s}}{s}(\frac{n}{1})^2 \in S^{-1}Q$. Then, there exists $\tilde{s} \in S$ such that $\tilde{s}\acute{s}m^2 \in Q$ or $\tilde{s}\acute{s}n^2 \in Q$. Because of $Z_p(R) \cap S = \emptyset$, we have $\acute{s}m^2 \in Q$ or $\acute{s}n^2 \in Q$ for some $\acute{s} \in \acute{S}$. □

Proposition 3. *Let $f : R \to \widetilde{R}$ be a ring homomorphism and S be an m.c.s of R. Then,*

1. *If f is an epimorphism, and Q is an S-2-prime ideal of R containing $ker(f)$, then $f(Q)$ is an $f(S)$-2-prime ideal of $\widetilde{R}$.*
2. *If $\widetilde{Q}$ is an $f(S)$-2-prime ideal of $\widetilde{R}$ where $f(S)$ does not contain zero, then $f^{-1}(\widetilde{Q})$ is an S-2-prime ideal of R.*

Proof. 1. It is clear that $f(S)$ is an m.c.s of $\widetilde{R}$ because $f(S)$ does not contain zero. Assume that $t \in f(S) \cap f(Q)$; that is $t = f(q) = f(s)$ for some $q \in Q$ and $s \in S$. Then, $s - q \in ker(f) \subseteq Q$ and so $s \in Q \cap S$, which is a contradiction. Hence, $f(S) \cap f(Q) = \emptyset$. Now, suppose that $\acute{m}\acute{n} \in f(Q)$ for some $\acute{m}, \acute{n} \in \widetilde{R}$. Then, there are $m, n \in R$ such that $f(m) = \acute{m}$ and $f(n) = \acute{n}$. Then, $mn \in Q$ and there exists $s \in S$ such that $sm^2 \in Q$ or $sn^2 \in Q$. This implies that $f(s)(\acute{m})^2 \in f(Q)$ or $f(s)(\acute{n})^2 \in f(Q)$, as needed.

2. We omit the proof as it is clear. □

The following corollary is a direct consequence of Proposition 3, because we can consider $f : R \to R/P$ as an epimorphism and the natural injection $i : R \to \widetilde{R}$, respectively.

Corollary 1. *Let Q be a proper ideal of R containing an ideal P of R.*

1. *Q is an S-2-prime ideal of R if and only if Q/P is an $\overline{S}$-2-prime ideal of R/P.*
2. *If R is a subring of $\widetilde{R}$ and Q is an S-2-prime ideal of $\widetilde{R}$, then $Q \cap R$ is an S-2-prime ideal of R.*

It is seen easily that if Q is an S-2-prime ideal of R, then $\sqrt{Q} = P$ is an S-prime ideal of R. So, we can say that Q is a P-S-2-prime ideal of R. Now, we will give a theorem analogous to the prime avoidance lemma.

Theorem 4. *Let S be an m.c.s of R and Q be an ideal of R. Suppose that at least n-2 of J_i is a P_i-S-2-prime ideal of R for $i \in \{1, 2, \ldots, n\}$ and $Q \subseteq \cup_{i=1}^{n} J_i$, then $sQ \subseteq P_k$ for some $k \in \{1, 2, \ldots, n\}$ and $s \in S$.*

Proof. Suppose that $Q \subseteq \cup_{i=1}^{n} J_i$ where at least n-2 of $J_1, J_2, \ldots, J_n$ are S-2-prime ideals of R. Since $Q \subseteq \cup_{i=1}^{n} J_i \subseteq \cup_{i=1}^{n} (P_i : s_i)$ and at least n-2 of $(P_i : s_i)$ are prime ideals by [2]. Then, the prime avoidance lemma implies that there exists $k \in \{1, 2, \ldots, n\}$ such that $Q \subseteq (P_k : s_k)$. Thus, we conclude that $s_k Q \subseteq P_k$. □

Let $n \in \mathbb{N}$. For any prime p dividing n, we have the m.c.s $S_p = \{1, \bar{p}, \bar{p}^2, \bar{p}^3, \ldots\}$ of $\mathbb{Z}_n$. In the following, we totally determine S_p-2-prime ideals of $\mathbb{Z}_n$ for any p dividing n.

Theorem 5. *Let $n \in \mathbb{N}$.*

1. *If $n = p^r$ for some prime integer p and $r \geq 1$, then $\mathbb{Z}_n$ has no S_p-2-prime ideals.*
2. *If $n = p_1^{r_1} p_2^{r_2}$ where p_1 and p_2 are distinct prime integers and $r_1, r_2 \geq 1$, then every ideal of $\mathbb{Z}_n$ disjoint with S_{p_i} is an S_{p_i}-2-prime ideal for all $i = 1, 2$.*
3. *If $n = p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}$ where $p_1, p_2, \ldots, p_k$ are distinct prime integers and $k \geq 3$, then $\mathbb{Z}_n$ has no S_{p_i}-2-prime ideals for all $i = 1, 2, \ldots, k$.*

Proof. 1. Let $n = p^r$. Then, $\mathbb{Z}_n$ has no S_p-2-prime ideal as $I \cap S_p \neq \emptyset$.

2. Let $Q = \left\langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} \right\rangle$ be an ideal of $\mathbb{Z}_n$ disjoint with S_{p_1}. Then, we must have $t_2 \geq 1$. Set $m := \begin{cases} t_2/2 & , \text{ if } t_2 \text{ is even} \\ (t_2+1)/2 & , \text{ if } t_2 \text{ is odd} \end{cases}$ and $s = \bar{p}_1^{t_1} \in S_{p_1}$. Suppose that $\alpha\beta \in Q$ for $\alpha, \beta \in \mathbb{Z}_n$. If $\alpha \in \langle \bar{p}_2^m \rangle$, then $s\alpha^2 \in Q$. If $a \notin \langle \bar{p}_2 \rangle$, then clearly $\beta \in \langle \bar{p}_2^m \rangle$ and so $s\beta^2 \in Q$. Thus, Q is an S_{p_1}-2-prime ideal of $\mathbb{Z}_n$. Similarly, every ideal of $\mathbb{Z}_n$ distinct with S_{p_2} is an S_{p_2}-2-prime ideal.

3. Let $Q = \left\langle \bar{p}_1^{t_1} \bar{p}_2^{t_2} \ldots \bar{p}_k^{t_k} \right\rangle$ be an ideal of $\mathbb{Z}_n$ distinct with S_{p_1}. Then, there exists $j \neq 1$ such that $t_j \geq 1$; say $j = k$. Thus, $\bar{p}_k^{t_k}(\bar{p}_1^{t_1} \bar{p}_2^{t_2} \ldots \bar{p}_{k-1}^{t_{k-1}}) \in Q$ but $s(\bar{p}_k^{t_k})^2 \notin Q$ and $s(\bar{p}_1^{t_1} \bar{p}_2^{t_2} \ldots \bar{p}_{k-1}^{t_{k-1}})^2 \notin Q$ for all $s \in S_{p_1}$. So, Q is not an S_{p_1}-2-ideal of $\mathbb{Z}_n$ and similarly, Q is not an S_{p_i}-2-ideal of $\mathbb{Z}_n$ for all $i = 1, 2, \ldots, k$. □

3. *S*-2-Prime Ideals in Idealization and Amalgamation Rings

Let R be a commutative ring with identity and M be a unitary R-module. The trivial extension or idealization of R in M is a commutative ring $R(+)M = \{(\widetilde{r}, m) : \widetilde{r} \in R, m \in M\}$ with usual addition and the multiplication $(\widetilde{r_1}, m_1)(\widetilde{r_2}, m_2) = (\widetilde{r_1}\widetilde{r_2}, \widetilde{r_1}m_2 + \widetilde{r_2}m_1)$ for all $(\widetilde{r_1}, m_1), (\widetilde{r_2}, m_2) \in R(+)M$ [12,13]. It is clear that if S is an m.c.s of R, then $S(+)0$ and $S(+)M$ are m.c.ss of $R(+)M$.

Theorem 6. *Assume that R and M are as the above, S is an m.c.s of R, and Q is a proper ideal of R with $Q \cap S = \emptyset$. Then, $Q(+)M$ is an $(S(+)0)$-2-prime ideal (resp. $(S(+)M)$-2-prime ideal) of $R(+)M$ if and only if Q is an S-2-prime ideal of R.*

Proof. It is explicit that $(S(+)0) \cap (Q(+)M) = \emptyset \Leftrightarrow S \cap Q = \emptyset$. Similarly, $(S(+)M) \cap (Q(+)M) = \emptyset \Leftrightarrow S \cap Q = \emptyset$. Let $\alpha\beta \in Q$ for some $\alpha, \beta \in R$. We have $(\alpha, 0)(\beta, 0) \in Q(+)M$. Then, there exists $(s, 0) \in (S(+)0)$ (and there exists $(s, m) \in (S(+)M)$) such that $(s, 0)(\alpha, 0)^2 \in Q(+)M$ or $(s, 0)(\beta, 0)^2 \in Q(+)M$ (resp. $(s, m)(\alpha, 0)^2 = (s\alpha^2, m\alpha^2) \in Q(+)M$ or $(s, m)(\beta, 0)^2 = (s\beta^2, m\beta^2) \in Q(+)M$). Hence, $s\alpha^2 \in Q$ or $s\beta^2 \in Q$; and so,

Q is an S-2-prime ideal of R. Conversely, suppose that $(\alpha,\beta)(\gamma,\sigma) \in Q(+)M$ for some (α,β), $(\gamma,\sigma) \in R(+)M$. We have $\alpha\gamma \in Q$, and there exists $s \in S$ such that $s\alpha^2 \in Q$ or $s\gamma^2 \in Q$. Thus, $(s,0)(\alpha,\beta)^2 = (s\alpha^2, 2s\alpha\beta) \in Q(+)M$ or $(s,0)(\gamma,\sigma)^2 = (s\gamma^2, 2s\gamma\sigma) \in Q(+)M$ (resp. $(s,m)(\alpha,\beta)^2 = (s\alpha^2, 2s\alpha\beta + m\alpha^2) \in Q(+)M$ or $(s,m)(\gamma,\sigma)^2 = (s\gamma^2, 2s\gamma\sigma + m\gamma^2) \in Q(+)M$). We conclude that $Q(+)M$ is an $(S(+)0)$-2-prime ideal (resp. $(S(+)M)$-2-prime ideal) of $R(+)M$. □

Let A and B be commutative rings with identity, J be an ideal of B and $f : A \to B$ be a homomorphism. Then, $A \bowtie^f J = \{(a, f(a)+j) : a \in A, j \in J\}$ is called the amalgamation of A with B along the ideal J with regard to f [15]. For an m.c.s S of A, take $(S \bowtie^f 0) = \{(s, f(s)) : s \in S\}$. Then, $(S \bowtie^f 0)$ is an m.c.s of $A \bowtie^f J$. Moreover, if $f(S)$ does not contain zero, then $f(S)$ is an m.c.s of B. Assume that Q is an ideal of A, and P is an ideal of $f(A)+J$. We know that $Q \bowtie^f J = \{(q, f(q)+j) : q \in Q, j \in J\}$ and $P^f = \{(a, f(a)+j) : a \in A, j \in J, f(a)+j \in P\}$ are ideals of $A \bowtie^f J$ [4,14].

Theorem 7. *Assume that the above amalgamation property is held. Let S be an m.c.s of A and Q be a proper ideal of A, and P be a proper ideal of $f(A)+J$.*

1. *$Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$ if and only if Q is an S-2-prime ideal of A.*
2. *If $f(S)$ does not contain zero, then P^f is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$ if and only if P is an $f(S)$-2-prime ideal of $f(A)+J$.*

Proof. 1. It is explicit that $(S \bowtie^f 0) \cap (Q \bowtie^f J) = \emptyset \Leftrightarrow S \cap Q = \emptyset$. Assume that $\alpha\beta \in Q$ for some α, $\beta \in A$. Then, $(\alpha, f(\alpha))(\beta, f(\beta)) = (\alpha\beta, f(\alpha\beta)) \in Q \bowtie^f J$ and so there exists $(s, f(s)) \in (S \bowtie^f 0)$ such that $(s, f(s))(\alpha, f(\alpha))^2 \in Q \bowtie^f J$ or $(s, f(s))(\beta, f(\beta))^2 \in Q \bowtie^f J$. It refers to the fact that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$, and so, Q is an S-2-prime ideal of A. Conversely, let $(\alpha, f(\alpha)+j)$, $(\beta, f(\beta)+k) \in A \bowtie^f J$ with $(\alpha, f(\alpha)+j)(\beta, f(\beta)+k) \in Q \bowtie^f J$. We have $\alpha\beta \in Q$, and there exists $s \in S$ such that $s\alpha^2 \in Q$ or $s\beta^2 \in Q$. If $s\alpha^2 \in Q$, then $(s, f(s))(\alpha, f(\alpha)+j)^2 = (s\alpha^2, f(s\alpha^2) + 2f(s\alpha)j + f(s)j^2) \in Q \bowtie^f J$. Similarly, if $s\beta^2 \in Q$, then $(s, f(s))(\beta, f(\beta)+k)^2 \in Q \bowtie^f J$. Therefore, $Q \bowtie^f J$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$.

2. Clearly, $f(S) \cap P = \emptyset \Leftrightarrow (S \bowtie^f 0) \cap P^f = \emptyset$. Let α, $\beta \in f(A)+J$ with $\alpha\beta \in P$. Then, $\alpha = f(\alpha_1) + j_1$ and $\beta = f(\alpha_2) + j_2$ for some α_1, $\alpha_2 \in A$ and j_1, $j_2 \in J$. Hence, $(\alpha_1, f(\alpha_1)+j_1)(\alpha_2, f(\alpha_2)+j_2) = (\alpha_1\alpha_2, (f(\alpha_1)+j_1)(f(\alpha_2)+j_2)) \in P^f$. Then, there exists $s \in S$ such that $(s, f(s))(\alpha_1, f(\alpha_1)+j_1)^2 \in P^f$ or $(s, f(s))(\alpha_2, f(\alpha_2)+j_2)^2 \in P^f$. That is, $f(s)(f(\alpha_1)+j_1)^2 \in P$ or $f(s)(f(\alpha_2)+j_2)^2 \in P$, and thus, P is an $f(S)$-2-prime ideal of $f(A)+J$. Conversely, let $(\alpha, f(\alpha)+j_1)(\beta, f(\beta)+j_2) \in P^f$. Then, $(f(\alpha)+j_1)(f(\beta)+j_2) \in P$ and there exists $f(s) \in f(S)$ such that $f(s)(f(\alpha)+j_1)^2 \in P$ or $f(s)(f(\beta)+j_2)^2 \in P$. Therefore, $(s, f(s))(\alpha, f(\alpha)+j_1)^2 = (s\alpha^2, f(s)(f(\alpha)+j_1)^2) \in P^f$ or $(s, f(s))(\beta, f(\beta)+j_2)^2 = (s\beta^2, f(s)(f(\beta)+j_2)^2) \in P^f$. □

Proposition 4. *Assume that $f : A \to B$ is a ring homomorphism, and J is an ideal of B. Let Q is a proper ideal of A and K is an ideal of $f(A)+J$ with $f(Q)J \subseteq K \subseteq J$. If $Q \bowtie^f K = \{(q, f(q)+k) : q \in Q, k \in K\}$ is an $(S \bowtie^f 0)$-2-prime ideal of $A \bowtie^f J$, then Q is an S-2-prime ideal of A.*

Proof. From [4], we know that $Q \bowtie^f K$ is an ideal of $A \bowtie^f J$. Let α, $\beta \in A$ with $\alpha\beta \in Q$. Hence, $(\alpha, f(\alpha))(\beta, f(\beta)) = (\alpha\beta, f(\alpha\beta)) \in Q \bowtie^f K$. Because of assumption, there exists $(s, f(s)) \in (S \bowtie^f 0)$ such that $(s, f(s))(\alpha, f(\alpha))^2 \in Q \bowtie^f K$ or $(s, f(s))(\beta, f(\beta))^2 \in Q \bowtie^f K$. Hence, $s\alpha^2 \in Q$ or $s\beta^2 \in Q$ and so Q is an S-2-prime ideal of A. □

Let I be a proper ideal of R. The amalgamated duplication of R along I is described by $R \bowtie I = \{(\tilde{r}, \tilde{r}+i) : \tilde{r} \in R, i \in I\}$ [14]. We may have the following corollaries because of Theorem 7, (1.) and Proposition 4.

Corollary 2. *Assume that I, Q are proper ideals of R and S is an m.c.s of R. Then, $Q \bowtie I$ is an $(S \bowtie 0)$-2-prime ideal of $R \bowtie I$ if and only if Q is an S-2-prime ideal of R.*

Corollary 3. *Assume that Q, K, J are ideals of A, where Q is proper ideal with $QJ \subseteq K \subset J$. If $Q \bowtie K$ is an $(S \bowtie 0)$-2-prime ideal of $A \bowtie J$, then Q is an S-2-prime ideal of A.*

4. S-Properties of Compactly Packedness, Compactly 2-Packedness and Coprimely Packedness on Trivial Extension

A ring R is called a compactly packed ring if whenever $Q \subseteq \cup_{i\in\wedge}P_i$ for some ideal Q of R and a family of prime ideals $\{P_i\}_{i\in\wedge}$ of R, then there exists $k \in \wedge$ such that $Q \subseteq P_k$ [16]. Smith [17] proved that R is a compactly packed ring if and only if $P \subseteq \cup_{i\in\wedge}P_i$ for some prime ideal P of R and a family of prime ideals $\{P_i\}_{i\in\wedge}$ of R implies that $P \subseteq P_k$ for some $k \in \wedge$. On the other hand, Erdoğdu [18] presented the notion of coprimely packed rings, a generalization of compactly packed rings. A ring R is called a coprimely packed ring if whenever $Q + P_i = R$ for some ideal Q of R and a family of prime ideals $\{P_i\}_{i\in\wedge}$ of R, then $Q \nsubseteq \cup_{i\in\wedge}P_i$. Note that every compactly packed ring is a coprimely packed ring. The converse of this indication is correct when R is a domain with a Krull dimension is one Proposition 2.2 [18].

In this section, we study the union of S-prime and S-2-prime ideals in commutative rings.

Definition 2. *Let R be a ring and S be an m.c.s of R. Then,*

1. *R is called a compactly S-packed ring if $Q \subseteq \cup_{i\in\wedge}P_i$, where Q is an ideal of R and $\{P_i\}_{i\in\wedge}$ is a family of S-prime ideals of R, then there exists $k \in \wedge$ and $s \in S$ such that $sQ \subseteq P_k$.*
2. *R is called a compactly S-2-packed ring if $Q \subseteq \cup_{i\in\wedge}P_i$, where Q is an ideal of R and $\{P_i\}_{i\in\wedge}$ is a family of S-2-prime ideals of R, then there exists $k \in \wedge$ and $s \in S$ such that $sQ \subseteq P_k$.*

Note that as every S-prime ideal is an S-2-prime ideal, clearly every compactly S-2-packed ring is a compactly S-packed ring. However, the converse statement is not true in general.

Example 4. *Let $R = \mathbb{Z}_3[X, Y, Z]/Q$, where $Q = (X^2, Y^2, Z^2, XY, XZ, YZ)$ and $S = \overline{1}$ an m.c.s of R. Then, R is a local ring with unique maximal ideal $M = (X, Y, Z)/Q = \sqrt{(\overline{0})}$ and $M^2 = (\overline{0})$. From Lemma 1 [7], we have that every proper ideal of R is a 2-prime ideal, so an S-2-prime ideal. In addition, $|specR| = 1$ and R is a compactly S-packed ring. Now, consider the ideals $K = \{\overline{0}, \overline{x}, \overline{y}, \overline{x+y}\}$, $P_1 = \{\overline{0}, \overline{x}\}$, $P_2 = \{\overline{0}, \overline{y}\}$ and $P_3 = \{\overline{0}, \overline{x+y}\}$ of R. Since $K \subseteq \cup_{i=1}^{3}P_i$ and $sK \nsubseteq P_i$ for all $1 \leq i \leq 3$ and for $s = \overline{1} \in S$. It means that R is not a compactly S-2-packed ring.*

A ring R is called a von-Neumann regular ring if for every $\alpha \in R$, there exists $r \in R$ such that $\alpha = \alpha^2 r$ [22]. Moreover, it is shown that in a von-Neumann regular ring $Q = \sqrt{Q}$ Theorem 1 [21]. Furthermore, note that if R is a von-Neumann regular ring then every S-2-prime ideal coincides with S-prime ideal of R where S is an m.c.s of R. Thus, we conclude the following corollary.

Corollary 4. *Suppose that R is a von-Neumann regular ring and S is an m.c.s of R. Then, R is a compactly S-packed ring if and only if R is a compactly S-2-packed ring.*

Let Q be an ideal of R and N be a submodule of an R-module M. Then, $Q(+)N$ is an ideal of $R(+)M$ if and only if $QM \subseteq N$. In this case, $Q(+)N$ is a called homogeneous ideal of $R(+)M$. In [13], authors described the conditions under which every ideal of $R(+)M$ is a homogeneous ideal. Additionally, $Q(+)M$ is always a homogeneous ideal.

Now, we will examine the conditions under which the trivial extension or idealization $R(+)M$ of an R-module M is a compactly $(S(+)0)$-packed ring and compactly $(S(+)0)$-2-packed ring (resp. compactly $(S(+)M)$-packed ring and compactly $(S(+)M)$-2-packed ring). From Theorem 3.2 [13], every prime ideal $R(+)M$ has the form $P = p(+)M$ for some prime ideal p of R. We know that in trivial extension, if S is an m.c.s of R, then

$(S(+)0)$ and $(S(+)M)$ are m.c.ss of $R(+)M$. Moreover, we can verify that every $(S(+)0)$ (and $(S(+)M)$)-prime ideal of $R(+)M$ has the form $P = p(+)M$ for some S-prime ideal p of R since every prime ideal disjoint with S is an S-prime ideal.

Lemma 2. *Let R be a commutative ring, S be an m.c.s of R and M be an R-module. The $S(+)0$ (or $S(+)M$)-prime (resp. $S(+)0$ (or $S(+)M$)-2-prime) ideals of $R(+)M$ has the form $p(+)M$ where p is an S-prime (resp. S-2-prime) ideal of R.*

Proof. From Theorem 3.1 [13], the ideals of $R(+)M$ containing $0(+)M$ are of the form $p(+)M$ for some ideal p of R. Note that $(p(+)M) \cap (S(+)0) = \emptyset$ (or $(p(+)M) \cap (S(+)M) = \emptyset$) if and only if $p \cap S = \emptyset$. Suppose that $p(+)M$ is an $S(+)0$ (or $S(+)M$)-prime (resp. $S(+)0$ (or $S(+)M$)-2-prime) ideal of $R(+)M$. Let $\alpha, \beta \in R$ with $\alpha\beta \in p$. Then, $(\alpha, 0)(\beta, 0) \in p(+)M$ which implies that there exists $\acute{s} = (s, 0) \in S(+)0$ ($\acute{s} = (s, m) \in S(+)M$) such that $\acute{s}(\alpha, 0) \in p(+)M$ or $\acute{s}(\beta, 0) \in p(+)M$ (resp. $\acute{s}(\alpha, 0)^2 \in p(+)M$ or $\acute{s}(\beta, 0)^2 \in p(+)M$). Hence, we have $s\alpha \in p$ or $s\beta \in p$ (resp. $s\alpha^2 \in p$ or $s\beta^2 \in p$), so p is an S-prime (resp. S-2-prime) ideal of R. The converse part is straightforward. □

Theorem 8. *Suppose that M is an R-module and S is an m.c.s of R. Then, the following statements are equivalent:*

1. *R is a compactly S-packed ring.*
2. *$R(+)M$ is a compactly $(S(+)0)$-packed ring (and compactly $(S(+)M)$-packed ring).*

Proof. 1.$\Longrightarrow$2. Let R be a compactly S-packed ring. Suppose that $Q \subseteq \cup_{i\in\wedge} P_i$ for some ideal Q of $R(+)M$ and a family of $(S(+)0)$-prime ideals (and $(S(+)M)$-prime ideals) $\{P_i\}_{i\in\wedge}$ of $R(+)M$. Then, $Q = q(+)M$ and $P_i = p_i(+)M$ for some ideal q and S-prime ideal p_i of R by the Lemma 2. We have $q \subseteq \cup_{i\in\wedge} p_i$. Since R is a compactly S-packed ring, there exists $s \in S$ such that $sq \subseteq p_i$ for some $i \in \wedge$. There exists $(s, 0) \in (S(+)0)$ (resp. there exists $(s, m) \in (S(+)M)$) such that

$$(s,0)Q = (s,0)(q(+)M) \subseteq p_i(+)M = P_i$$
$$(\text{resp. } (s,m)Q = (s,m)(q(+)M) \subseteq p_i(+)M = P_i)$$

Thus, $R(+)M$ is a compactly $(S(+)0)$-packed ring (and compactly $(S(+)M)$-packed ring).

2.$\Longrightarrow$1. Let $R(+)M$ be a compactly $(S(+)0)$-packed ring (and compactly $(S(+)M)$-packed ring). Suppose that $Q \subseteq \cup_{i\in\wedge} p_i$ for some ideal Q of R and a family of S-prime ideals $\{p_i\}_{i\in\wedge}$ of R. Then, $Q(+)M \subseteq \cup_{i\in\wedge}(p_i(+)M)$ where $(p_i(+)M)$ is $(S(+)0)$-prime ideal (resp. $(S(+)M)$-prime ideal) of $R(+)M$ for all $i \in \wedge$. Since $R(+)M$ is a compactly $(S(+)0)$-packed ring (and compactly $(S(+)M)$-packed ring), there exists $(s, 0) \in (S(+)0)$ (resp. there exists $(s, m) \in (S(+)M)$) such that

$$(s,0)(Q(+)M) \subseteq p_i(+)M$$
$$(\text{resp. } (s,m)(Q(+)M) \subseteq p_i(+)M)$$

for some $i \in \wedge$. This implies that $sQ \subseteq p_i$ for some $i \in \wedge$ and $s \in S$. □

Theorem 9. *Let S be an m.c.s of a ring R and M be an R-module. If $R(+)M$ is a compactly $(S(+)0)$-2-packed ring (and compactly $(S(+)M)$-2-packed ring), then R is a compactly S-2-packed ring.*

Proof. Let $Q \subseteq \cup_{i\in\wedge} P_i$ where Q is an ideal of R and $\{P_i\}_{i\in\wedge}$ is a family of S-2-prime ideals of R. From Lemma 2, $(P_i(+)M)$ is $(S(+)0)$ (and $(S(+)M)$)-2-prime ideal of $R(+)M$ for all $i \in \wedge$. Clearly, $Q(+)M \subseteq \cup_{i\in\wedge}(P_i(+)M)$. Since $R(+)M$ is a compactly $(S(+)0)$-2-packed ring (and compactly $(S(+)M)$-2-packed ring), there exists $i \in \wedge$ and $(s,0) \in (S(+)0)$ (resp. $(s,m) \in (S(+)M)$) such that

$$(s,0)(Q(+)M) \subseteq (P_i(+)M)$$
$$(\text{resp. } (s,m)(Q(+)M) \subseteq (P_i(+)M))$$

This implies that $sQ \subseteq P_i$ for some $s \in S$ and $i \in \wedge$. Therefore, R is a compactly S-2-packed ring. □

In the Theorem 9, the reverse statement is not correct. Consider the next example.

Example 5. *Suppose that* $R = \mathbb{Z}$ *as a ring,* $S = \{1\}$ *as an m.c.s of* R *and* $M = \mathbb{Z}_3[X, Y, Z]/Q$ *where* $Q = (X^2, Y^2, Z^2, XY, XZ, YZ)$ *as in the Example 4. Since* R *is a principal ideal ring, clearly* R *is a compactly* 2*-packed ring. Because of* $S = \{1\}$*,* R *is a compactly* S*-2-packed ring. Consider the ring* $R(+)M$ *and the ideal* $J = (0)(+)N$ *where* N *is a submodule of* M*. Then,* J *is an* $(S(+)0)$ *(and* $(S(+)M)$*)-2-prime ideal of* $R(+)M$*. In order to prove this, let* $(x,m)(y,\acute{m}) = (xy, x\acute{m} + ym) \in J$ *for some* $x, y \in \mathbb{Z}$ *and* $m, \acute{m} \in M$*. We have* $xy = 0$*. Without loss of generality, take* $x = 0$*. Then, the following statements are provided for some* $(s,0) \in (S(+)0)$ *(resp. for some* $(s,\widetilde{m}) \in (S(+)M)$*).*

$$(s,0)(x,m)^2 = (s,0)(0,m)^2 = (0,0) \in J$$
$$(\text{resp. } (s,\widetilde{m})(x,m)^2 = (s,\widetilde{m})(0,m)^2 = (0,0) \in J)$$

Hence, $J = (0)(+)N$ *is an* $(S(+)0)$ *(and* $(S(+)M)$*)-2-prime ideal of* $R(+)M$*. Take* $K = \{\overline{0}, \overline{x}, \overline{y}, \overline{x+y}\}$*,* $P_1 = \{\overline{0}, \overline{x}\}$*,* $P_2 = \{\overline{0}, \overline{y}\}$ *and* $P_3 = \{\overline{0}, \overline{x+y}\}$ *as in Example 4. Clearly,* K*,* P_1*,* P_2 *and* P_3 *are submodules of* M*. Consider* $\acute{K} = (0)(+)K$*,* $\acute{P_1} = (0)(+)\ P_1$*,* $\acute{P_2} = (0)(+)\ P_2$ *and* $\acute{P_3} = (0)(+)\ P_3$*. We can achieve* $\acute{K}$*,* $\acute{P_1}$*,* $\acute{P_2}$ *and* $\acute{P_3}$ *are* $(S(+)0)$ *(and* $(S(+)M)$*)-2-prime ideals of* $R(+)M$*. Moreover,*

$$\acute{K} \subseteq \cup_{i=1}^{3} \acute{P_i} \text{ and } (s,0)\acute{K} \nsubseteq \acute{P_i} \text{ for all } (s,0) \in (S(+)0) \text{ and } 1 \leq i \leq 3$$
$$(\text{resp. } \acute{K} \subseteq \cup_{i=1}^{3} \acute{P_i} \text{ and } (s,m)\acute{K} \nsubseteq \acute{P_i} \text{ for all } (s,m) \in (S(+)M) \text{ and } 1 \leq i \leq 3).$$

Hence, $R(+)M$ *is not a compactly* $(S(+)0)$*-2-packed ring (and not compactly* $(S(+)M)$*-2-packed ring).*

Now, we will examine the coprimely S-packed rings for trivial extention.

Definition 3. *Let* S *be an m.c.s of a ring* R*. Then,* R *is called a coprimely* S*-packed ring if whenever* $Q + P_i \subseteq R$ *for some ideal* Q *of* R *and a family of* S*-prime ideals* $\{P_i\}_{i \in \wedge}$ *of* R*, then* $sQ \nsubseteq \cup_{i \in \wedge} P_i$ *for every* $s \in S$*.*

Theorem 10. *Suppose that* S *is an m.c.s of* R *and* M *is an* R*-module. Then,* $R(+)M$ *is a coprimely* $(S(+)0)$*-packed ring (and coprimely* $(S(+)M)$*-packed ring) if and only if* R *is a coprimely* S*-packed ring.*

Proof. $\Longrightarrow$: Let $R(+)M$ be a coprimely $(S(+)0)$-packed ring (and coprimely $(S(+)M)$-packed ring). We know that every factor ring of a coprimely packed ring is a coprimely packed ring from Remark 2 [18]. Moreover, every coprimely S-packed ring is a coprimely packed ring since every prime ideal disjoint with S is S-prime ideal. This implies that every factor ring of $R(+)M$ is a coprimely $(S(+)0)$-packed ring (and coprimely $(S(+)M)$-packed ring). Because of $(R(+)M)/((0)(+)M) \simeq R$, we have that R is a coprimely S-packed ring.

$\Longleftarrow$: Let R be a coprimely S-packed ring. Suppose that $Q + P_i = R(+)M$ for some ideal Q of $R(+)M$ and a family of $(S(+)0)$-prime (and $(S(+)M)$-prime) ideals $\{P_i\}_{i\in\wedge}$ of $R(+)M$. Then, we can write $P_i = p_i(+)M$ for some S-prime ideals p_i of R. Take

$$J = \{\alpha \in R : (\alpha, m) \in Q \text{ for some } m \in M\}.$$

It is easy to see that J is an ideal of R. Since $Q + P_i = R(+)M$ for all $i \in \wedge$, there exists $(\alpha, m) \in Q$ and $(\beta, \acute{m}) \in P_i$ such that $(\alpha, m) + (\beta, \acute{m}) = (1, 0)$ refers $\alpha + \beta = 1$. Hence, $J + p_i = R$. Because R is a coprimely S-packed ring, we obtain $sJ \nsubseteq \cup_{i\in\wedge} p_i$ for all $s \in S$. Take an element $\widetilde{z} \in sJ - \cup_{i\in\wedge} p_i$. From the description of J, there exists $\widetilde{m} \in M$ such that $(s, 0)(\widetilde{z}, \widetilde{m}) = (s\widetilde{z}, s\widetilde{m}) \in (s, 0)Q$ for all $(s, 0) \in (S(+)0)$ (and $(s, m)(\widetilde{z}, \widetilde{m}) = (s\widetilde{z}, s\widetilde{m} + \widetilde{z}m) \in (s, m)Q$ for all $(s, m) \in (S(+)M)$). Since $(s, 0)(\widetilde{z}, \widetilde{m}) \notin \cup_{i\in\wedge}(p_i(+)M) = \cup_{i\in\wedge} P_i$, which refers to $(s, 0)Q \nsubseteq \cup_{i\in\wedge} P_i$ for all $(s, 0) \in (S(+)0)$ (resp. $(s, m)(\widetilde{z}, \widetilde{m}) \notin \cup_{i\in\wedge}(p_i(+)M) = \cup_{i\in\wedge} P_i$, which refers to $(s, m)Q \nsubseteq \cup_{i\in\wedge} P_i$ for all $(s, m) \in (S(+)M)$).

Thus, $R(+)M$ is a coprimely $(S(+)0)$-packed ring (and coprimely $(S(+)M)$-packed ring). □

5. Discussion and Conclusions

Since prime ideals are important for many research areas such as commutative algebra, numerous authors have delved into their generalizations, yielding diverse findings. Some of these generalizations, as discussed in the introduction, shed light on our work. Our objective is to devise a broader concept encompassing both S-prime and 2-prime ideals. To accomplish this, we define a proper ideal on commutative rings with the help of multiplicatively closed subsets. We thoroughly examine the properties of S-2-prime ideals and explore their relations with other classes of ideals. Furthermore, we investigate the behavior of S-2-prime ideals within idealization and amalgamated rings.

Our study demonstrates that several results established in prior works, such as in [2,6,7] with [5] can be obtained for S-2-prime ideals. We conclude that S-2-prime ideals serve as a generalization of both 2-prime and S-prime ideals. However, it is essential to note that for 2-prime ideals to be considered S-2-prime ideals, the set S must consist of units. Additionally, we show that analogous outcomes can be derived from the study in [7] by examining the union of S-prime and S-2-prime ideals.

By investigating S-2 prime ideals, we have addressed inquiries regarding their algebraic structures and identified similarities with 2-prime and S-prime ideals. Consequently, this study has raised several unresolved questions for future exploration. We enumerate some of these below to encourage the reader. Exploring the algebraic structures associated with commuting maps could offer valuable insights. Interested readers may find inspiration in [19] to delve into this area. A study could be conducted to ascertain if analogous outcomes emerge for weakly S-2-prime ideals, which serve as a broader extension of S-2-prime ideals. Investigating the relationships between weakly S-2-prime and S-2-prime ideals as well as their relations with other ideals, presents an intriguing avenue for research. Additionally, delving into the algebraic properties of S-2-prime ideals by further extending their generalization using δ ideal expansion and ϕ ideal reduction functions could yield valuable insights into their behavior and characteristics.

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