Czechoslovak Mathematical Journal, 70 (145) (2020), 1079–1090

$(\delta, 2)$-PRIMARY IDEALS OF A COMMUTATIVE RING

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Received March 29, 2019. Published online April 17, 2020.

Abstract. Let R be a commutative ring with nonzero identity, let $\mathcal{I}(\mathcal{R})$ be the set of all ideals of R and $\delta\colon \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ an expansion of ideals of R defined by $I \mapsto \delta(I)$. We introduce the concept of $(\delta, 2)$-primary ideals in commutative rings. A proper ideal I of R is called a $(\delta, 2)$-primary ideal if whenever $a, b \in R$ and $ab \in I$, then $a^2 \in I$ or $b^2 \in \delta(I)$. Our purpose is to extend the concept of 2-ideals to $(\delta, 2)$-primary ideals of commutative rings. Then we investigate the basic properties of $(\delta, 2)$-primary ideals and also discuss the relations among $(\delta, 2)$-primary, δ-primary and 2-prime ideals.

Keywords: $(\delta, 2)$-primary ideal; 2-prime ideal; δ-primary ideal

MSC 2020: 13A15, 13F05, 05A15, 13G05

1. Introduction

In this paper, all rings are supposed to be commutative with nonzero identity. Let I be a proper ideal of a ring R and let $\mathcal{I}(\mathcal{R})$ denote the set of all ideals of R. The radical of I is defined by $\{a \in R\colon a^n \in I \text{ for some } n \in \mathbb{N}\}$, denoted by $\sqrt{I}$. Let J be an ideal of R. Then the ideal $(I : J)$ consists of $r \in R$ with $rJ \subseteq I$, that is, $(I : J) = \{r \in R\colon rJ \subseteq I\}$. In particular, $(I : x) = \{r \in R\colon rx \in I\}$. For any undefined notation or terminology, see [3], [7] or [10]. In [6], the authors introduced 2-prime ideals and gave the basic properties and some applications of the concept on valuation rings. A proper ideal I of R is called 2-prime if whenever $a, b \in R$ and $ab \in I$ then either $a^2 \in I$ or $b^2 \in I$. Then in [11], the authors introduced a new class of ideals which is between the 2-prime ideals and quasi primary ideals. A proper ideal I of R is called strongly quasi primary if whenever $a, b \in R$ and $ab \in I$ then either $a^2 \in I$ or $b^n \in I$ for some $n \in \mathbb{N}$.

Zhao in [12] introduced the concept of expansions of ideals and extended many results of prime and primary ideals to the new concept. He called a δ-primary ideal I

DOI: 10.21136/CMJ.2020.0146-19

of R if $ab \in I$ and $a \notin I$ for some $a, b \in R$ imply $b \in \delta(I)$. From [12], a function δ from $\mathcal{I}(\mathcal{R})$ to $\mathcal{I}(\mathcal{R})$ is an ideal expansion if it has the following properties: $I \subseteq \delta(I)$ and if $I \subseteq J$ for some ideals I, J of R, then $\delta(I) \subseteq \delta(J)$. For example, δ_0 is the identity function, where $\delta_0(I) = I$ for all ideals I of R, and δ_1 is defined by $\delta_1(I) = \sqrt{I}$. For other examples, consider the functions δ_+ and δ_* of $\mathcal{I}(\mathcal{R})$ defined with $\delta_+(I) = I + J$, where $J \in \mathcal{I}(\mathcal{R})$ and $\delta_*(I) = (I : P)$, where $P \in \mathcal{I}(\mathcal{R})$ for all $I \in \mathcal{I}(\mathcal{R})$, respectively (see [4]). Recently, δ-semiprimary ideals were studied in [5].

In this paper, we introduce the concept of $(\delta, 2)$-primary ideals of R which is an expansion of 2-prime ideals. We call a proper ideal I of R a $(\delta, 2)$-primary if $a, b \in R$ and $ab \in I$, then $a^2 \in I$ or $b^2 \in \delta(I)$. Then we give many results of the new structure. Among these results with related this concept: In Section 2, we set up the relations among 2-prime ideals, primary ideals, δ-primary ideals and $(\delta, 2)$-primary ideals in Proposition 1. Then it is shown that (see Theorem 1) a proper ideal I of R is $(\delta, 2)$-primary if and only if $KL \subseteq I$ for any ideals K and L of R implies that either $K_2 \subseteq I$ or $L_2 \subseteq \delta(I)$. By Corollary 1, we obtain that if $2!$ is a unit in R, then I is a $(\delta, 2)$-primary ideal of R if and only if $KL \subseteq I$ for any ideals K and L of R implies $K^2 \subseteq I$ or $L^2 \subseteq \delta(I)$. Proposition 3 gives that if I is a $(\delta, 2)$-primary ideal of R and $x \in R - I$ is an idempotent element, then $(I : x)$ is a $(\delta, 2)$-primary ideal of R. In Theorem 2, we compare irreducible ideals with $(\delta, 2)$-primary ideals. Theorem 4 shows that if I is a $(\delta, 2)$-primary ideal of R and $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, then $\sqrt{I}$ is a δ-primary ideal of R. Then in Theorem 5, we have that every proper principal ideal is a $(\delta, 2)$-primary ideal of R if and only if every proper ideal is a $(\delta, 2)$-primary ideal of R. In Theorem 7, we have that if R is a von Neumann regular ring (or Boolean ring), then every $(\delta, 2)$-primary ideal and δ-primary ideal of R coincide. Let R be a valuation ring with the quotient field K. Then a proper ideal I of R is a $(\delta, 2)$-primary ideal of R if and only if for every $a, b \in K$ with $ab \in I$ and $a^2 \notin I$, then $b^2 \in \delta(I)$ (see Theorem 8). In Section 3, we give many examples which show that the converses of some relations are not satisfied in general.

2. PROPERTIES OF $(\delta, 2)$-PRIMARY IDEALS

Throughout this paper, R denotes a commutative ring with nonzero identity and δ is an expansion function of $\mathcal{I}(\mathcal{R})$.

Definition 1. A proper ideal I of R is called a $(\delta, 2)$-primary ideal if whenever $x, y \in R$ and $xy \in I$ imply $x^2 \in I$ or $y^2 \in \delta(I)$.

Note that every prime, δ-primary, 2-prime ideal is a $(\delta, 2)$-primary ideal. Actually, we obtain the following diagram which gives the relations between $(\delta, 2)$-primary ideal and other classical ideals in the lattice of ideals $L(R)$:

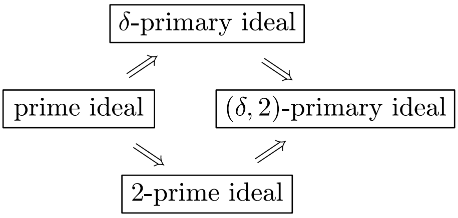

Figure 1. Relations between $(\delta, 2)$-primary ideals and other classical ideals

However, the converses of these relations are not satisfied in general, (see Example 4, Example 5 and Example 6). Since the following relations can be obtained by the definitions easily, we give our first result without proof.

Proposition 1. *Let I be a proper ideal of a commutative ring R. Then the following statements hold:*

(1) *I is a $(\delta_0, 2)$-primary ideal if and only if I is a 2-prime ideal.*

(2) *I is a $(\delta_1, 2)$-primary ideal if and only if I is a strongly quasi primary ideal.*

(3) *Let $\delta(I)$ be a prime ideal and $\delta(I)^2 \subseteq I$. Then I is a $(\delta, 2)$-primary ideal if and only if I is a 2-prime ideal.*

(4) *If I is a primary ideal, then I is a $(\delta_1, 2)$-primary ideal.*

(5) *If I is a 2-prime ideal, then I is a $(\delta, 2)$-primary ideal for every δ.*

(6) *If I is a δ-primary ideal, then I is a $(\delta, 2)$-primary ideal for every δ.*

(7) *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ with $\delta(\delta(I)) = \delta(I)$ for a proper ideal I of R. Then $\delta(I)$ is a 2-prime ideal of R if and only if $\delta(I)$ is a $(\delta, 2)$-primary ideal of R.*

The converses of (4), (5) and (6) do not hold in general, (see Example 2, Example 4, Example 5 and Example 6, respectively.)

Proposition 2. *Let I be a proper ideal of R and let δ, γ be two expansion functions of $\mathcal{I}(\mathcal{R})$ with $\delta(I) \subseteq \gamma(I)$. If I is a $(\delta, 2)$-primary ideal of R, then I is a $(\gamma, 2)$-primary ideal of R.*

Proof. It is clear. $\square$

The ideal generated by nth power of elements of a proper ideal I of R (i.e., $\{a^n\colon a \in I\}$) is denoted by I_n for a natural number n, see [1]. Recall that $I_n \subseteq I^n \subseteq I$. If $n!$ is a unit of R, we obtain $I_n = I^n$ by [1], Theorem 5.

Theorem 1. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I be a proper ideal of R. Then the following statements are equivalent:*

(1) *I is a $(\delta, 2)$-primary ideal of R.*

(2) *For every* $x \in R$, $(x) \subseteq (I : x)$ *or* $(I : x)_2 \subseteq \delta(I)$.
(3) $KL \subseteq I$ *for any ideals* K *and* L *of* R *implies* $K_2 \subseteq I$ *or* $L_2 \subseteq \delta(I)$.
(4) *For every* $x \in R$, $x^2 \in I$ *or* $(I : x)_2 \subseteq \delta(I)$.

Proof. (1) $\Rightarrow$ (2): Assume that I is a $(\delta, 2)$-primary ideal of R. If $x^2 \in I$ for any $x \in R$, then $(x) \subseteq (I : x)$. Suppose that $x^2 \notin I$. Let $y \in (I : x)$. Hence, we have $xy \in I$ and $x^2 \notin I$. Consequently, $y^2 \in \delta(I)$ and so we get $(I : x)_2 \subseteq \delta(I)$.

(2) $\Rightarrow$ (3): Assume that $KL \subseteq I$ and $K_2 \nsubseteq I$ for some ideals K and L of R. Then there is an element $k \in K$ with $k^2 \in K_2 - I$. Thus $k \notin (I : k)$. By (2), $(I : k)_2 \subseteq \delta(I)$. Then we have $kl \in KL \subseteq I$ for every $l \in L$. Then $l \in (I : k)$, that is, $l^2 \in (I : k)_2$. We obtain $L_2 \subseteq \delta(I)$ by our hypothesis.

(3) $\Rightarrow$ (4): Let $x^2 \notin I$. Take $y \in (I : x)$. Then $xy \in I$. Put $K = (x)$ and $L = (y)$ in (3). Since $K_2 \nsubseteq I$, we get $y^2 \in L_2 \subseteq \delta(I)$ by assumption. Thus $(I : x)_2 \subseteq \delta(I)$.

(4) $\Rightarrow$ (1): Let $xy \in I$ and $x^2 \notin I$ for some $x, y \in R$. Then $y \in (I : x)$. We get $y^2 \in (I : x)_2$. By our assumption it is clear that $y^2 \in \delta(I)$. ☐

We give the following results obtained by the previous theorem.

Corollary 1. *Let* δ *be an expansion function of* $\mathcal{I}(\mathcal{R})$, I *a proper ideal of* R *and* $2!$ *a unit in* R. *Then the following statements are equivalent:*
(1) I *is a* $(\delta, 2)$*-primary ideal of* R.
(2) $(x) \subseteq (I : x)$ *or* $(I : x)^2 \subseteq \delta(I)$ *for every* $x \in R$.
(3) $KL \subseteq I$ *for any ideals* K *and* L *of* R *implies* $K^2 \subseteq I$ *or* $L^2 \subseteq \delta(I)$.
(4) $x^2 \in I$ *or* $(I : x)^2 \subseteq \delta(I)$ *for every* $x \in R$.

Proposition 3. *Let* δ *be an expansion function of* $\mathcal{I}(\mathcal{R})$ *and* I *a proper ideal of* R. *If* I *is a* $(\delta, 2)$*-primary ideal of* R *and* $x \in R - I$ *is an idempotent element, then* $(I : x)$ *is a* $(\delta, 2)$*-primary ideal of* R.

Proof. Let $ab \in (I : x)$ and $a^2 \notin (I : x) = (I : x^2)$ for some $a, b \in R$. Then we have $abx \in I$ and $a^2x^2 \notin I$. By our assumption, $b^2 \in \delta(I) \subseteq \delta(I : x)$. Thus the proof is over. ☐

Theorem 2. *Let* δ *be an expansion function of* $\mathcal{I}(\mathcal{R})$, I *a proper ideal of* R *and* $(I : x) = (I : x^2)$ *for each* $x \in R - I$. *If* I *is an irreducible ideal, then* I *is a* $(\delta, 2)$*-primary ideal.*

Proof. Assume on the contrary that I is not a $(\delta, 2)$-primary ideal. Then there exist $a, b \in R$ with $ab \in I$ and neither $a^2 \in I$ nor $b^2 \in \delta(I)$. Then $a, b \notin I$ as $a^2 \notin I$ and $b^2 \notin \delta(I)$. Consider $(I + Ra) \cap (I + Rb)$. Clearly, $I \subseteq (I + Ra) \cap (I + Rb)$. Let $r \in (I + Ra) \cap (I + Rb)$. Then there are $x_1, x_2 \in I$ and $r_1, r_2 \in R$ with

$r = x_1 + r_1 a = x_2 + r_2 b$. As $x_2 b + r_2 b^2 = x_1 b + r_1 ab \in I$, we get $r_2 b^2 \in I$ and so we have $r_2 \in (I : b^2)$. By the assumption, we obtain $r_2 \in (I : b)$, that is, $r_2 b \in I$. Therefore, $r = x_2 + r_2 b \in I$, which contradicts our assumption that I is an irreducible ideal. Thus I is a $(\delta, 2)$-primary ideal. $\square$

Proposition 4. *If I is a $(\delta, 2)$-primary ideal of R and δ is an expansion function of R with $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, then $\sqrt{I}$ is a δ-primary ideal of R. In particular, if I is $(\delta_1, 2)$-primary, then $\sqrt{I}$ is a δ_1-primary ideal of R.*

Proof. Let $a, b \in R$ with $ab \in \sqrt{I}$ and $a \notin \sqrt{I}$. Then $a^n b^n \in I$ for some positive integer n. Since $a^{2n} \notin I$ and I is assumed to be a $(\delta, 2)$-ideal, we have $b^{2n} \in \delta(I)$. It means $b \in \sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, and we are done. $\square$

Recall that a proper ideal I of a commutative ring R is called semiprime if whenever $J^n \subset I$ for some ideal J of R and some positive integer n, then $J \subset I$. This means that $\sqrt{I} = I$. A prime ideal is always semiprime, but the converse part is not true. For example, an ideal (n) of $\mathbb{Z}$ is semiprime if and only if n is squarefree (for more information, see [8]). Then we get the following result when I is a semiprime ideal.

Proposition 5. *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a semiprime ideal of R. Then*

(1) *I is a 2-prime ideal of R if and only if I is prime.*

(2) *Let $\delta(I)$ be a semiprime ideal. Then I is a $(\delta, 2)$-primary ideal if and only if I is δ-primary.*

Recall from [12] that an ideal expansion δ of $\mathcal{I}(\mathcal{R})$ is said to be intersection preserving if it satisfies $\delta(I_1 \cap I_2 \cap \ldots \cap I_n) = \delta(I_1) \cap \delta(I_2) \cap \ldots \cap \delta(I_n)$ for any ideals $I_1, I_2, \ldots, I_n$ of R.

Proposition 6. *Let δ be an intersection preserving expansion function of $\mathcal{I}(\mathcal{R})$. If $I_1, I_2, \ldots, I_n$ are $(\delta, 2)$-primary ideals of R with $\delta(I_i) = P$ for all $i \in \{1, 2, \ldots, n\}$, then $\bigcap\limits_{i=1}^{n} I_i$ is a $(\delta, 2)$-primary ideal of R.*

Proof. Let $xy \in \bigcap\limits_{i=1}^{n} I_i$ and $x^2 \notin \bigcap\limits_{i=1}^{n} I_i$ for some $x, y \in R$. Then $x^2 \notin I_k$ for some $1 \leqslant k \leqslant n$. Thus $y^2 \in \delta(I_k) = P$ by our assumption. Thus $y^2 \in \delta\Big(\bigcap\limits_{i=1}^{n} I_i\Big)$ as $\delta\Big(\bigcap\limits_{i=1}^{n} I_i\Big) = \bigcap\limits_{i=1}^{n} \delta(I_i) = P$. $\square$

However, the intersection of two $(\delta, 2)$-primary ideals need not be $(\delta, 2)$-primary ideal (see Example 3).

Proposition 7. *Let* $\{I_i\colon i \in \Lambda\}$ *be a directed collection of* $(\delta, 2)$*-primary ideals of* R*. Then* $\bigcup\limits_{i\in\Lambda} I_i$ *is a* $(\delta, 2)$*-primary ideal of* R*.*

Proof. Let $ab \in \bigcup\limits_{i\in\Lambda} I_i$ with $a^2 \notin \bigcup\limits_{i\in\Lambda} I_i$. Since $ab \in I_j$ for some $j \in \Lambda$ and $a^2 \notin I_j$, it implies that $b^2 \in \delta(I_j) \subseteq \delta\Bigl(\bigcup\limits_{i\in\Lambda} I_i\Bigr)$, we are done. □

Let R and S be commutative rings with $1 \neq 0$ and let δ, γ be two expansion functions of $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}(\mathcal{S})$, respectively. Then a ring homomorphism $f\colon R \to S$ is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for all ideals I of S. Let γ_1 a radical operation on ideals of S and δ_1 a radical operation on ideals of R. A homomorphism from R to S is an example of $\delta_1\gamma_1$-homomorphism. Additionally, if f is a $\delta\gamma$-epimorphism and I is an ideal of R containing $\ker(f)$, then $\gamma(f(I)) = f(\delta(I))$, see [4].

Theorem 3. *Let* $f\colon R \to S$ *be a* $\delta\gamma$*-homomorphism, where* δ *and* γ *are expansion functions of* $\mathcal{I}(\mathcal{R})$ *and* $\mathcal{I}(\mathcal{S})$*, respectively. Then the following statements hold:*

(1) *If* J *is a* $(\gamma, 2)$*-primary ideal of* S*, then* $f^{-1}(J)$ *is a* $(\delta, 2)$*-primary ideal of* R*.*

(2) *Let* f *be an epimorphism and* I *a proper ideal of* R *with* $\ker(f) \subseteq I$*. Then* I *is* $(\delta, 2)$*-primary ideal of* R *if and only if* $f(I)$ *is a* $(\gamma, 2)$*-primary ideal of* S*.*

Proof. (1) Let $xy \in f^{-1}(J)$ for some $x, y \in R$. Then $f(xy) = f(x)f(y) \in J$, which implies $(f(x))^2 \in J$ or $(f(y))^2 \in \gamma(J)$. Then $f(x^2) \in J$ or $f(y^2) \in \gamma(J)$. Thus we have $x^2 \in f^{-1}(J)$ or $y^2 \in f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$ since f is a $\delta\gamma$-homomorphism. Thus $f^{-1}(J)$ is a $(\delta, 2)$-primary ideal of R.

(2) Let $xy \in f(I)$ for some $x, y \in S$. Then there are two elements $a, b \in I$ such that $x = f(a)$ and $y = f(b)$. Then $f(a)f(b) = f(ab) \in f(I)$ and since $\ker(f) \subseteq I$, we conclude $ab \in I$. We get $a^2 \in I$ or $b^2 \in \delta(I)$. Thus $f(a^2) = x^2 \in f(I)$ or $f(b^2) = y^2 \in f(\delta(I)) = \delta(f(I))$. Thus $f(I)$ is a $(\gamma, 2)$-primary ideal of S. □

Remark 1. Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I a proper ideal of R. Then the function $\delta_q\colon R/I \to R/I$, defined by $\delta_q(J/I) = \delta(J)/I$ for all ideals $I \subseteq J$, becomes an expansion function of R/I, see [4]. Consider the natural homomorphism $\pi : R \to R/J$. Then for ideals I of R with $\ker(\pi) \subseteq I$, we have $\delta_q(\pi(I)) = \pi(\delta(I))$.

Corollary 2. *Let* δ *be an expansion function of* $\mathcal{I}(\mathcal{R})$*, and let* I *and* J *be ideals of* R *with* $I \subseteq J$*. Then the following statements hold:*

(1) J *is a* $(\delta, 2)$*-primary ideal of* R *if and only if* J/I *is a* $(\delta_q, 2)$*-primary ideal of* R/I*.*

(2) *If* I *is a* $(\delta, 2)$*-primary ideal of* R *and* R' *is a subring with* $R' \nsubseteq I$*, then* $I \cap R'$ *is a* $(\delta, 2)$*-primary ideal of* R'*.*

Proof. (1) and (2) are clear. □

Let δ be an expansion function of ideals of a polynomial ring $R[X]$ where X is an indeterminate. Then observe that the function as in Remark 1, $\delta_q\colon R[X]/(X) \to R[X]/(X)$ defined by $\delta_q(J/(X)) = \delta(J)/(X)$ for all ideals J of $R[X]$ with $(X) \subseteq J$, is an expansion function of ideals of R as $R[X]/(X) \cong R$. According to these expansions, we have the equivalent situations as follows:

Corollary 3. *Let I be a proper ideal of R. Then the following statements are equivalent:*
(i) *I is a $(\delta_q, 2)$-primary ideal of R.*
(ii) *(I, X) is a $(\delta, 2)$-primary ideal of $R[X]$.*

Proof. From Corollary 2 we conclude that (I, X) is a $(\delta, 2)$-primary ideal of $R[X]$ if and only if $(I, X)/(X)$ is a $(\delta_q, 2)$-primary ideal of $R[X]/(X)$. Since $(I, X)/(X) \cong I$ and $R[X]/(X) \cong R$, the result is obtained. □

Let S be a multiplicatively closed subset of a ring R and δ an expansion function of $\mathcal{I}(\mathcal{R})$. Note that δ_S is an expansion function of $\mathcal{I}(\mathcal{R}_S)$ such that $\delta_S(I_S) = (\delta(I))_S$. In the next theorem we investigate $(\delta_S, 2)$-primary ideals of the localization R_S.

Theorem 4. *Let δ be an expansion function of R and S a multiplicatively closed subset of R. If a proper ideal I of R is a $(\delta, 2)$-primary ideal with $I \cap S = \emptyset$, then I_S is a $(\delta_S, 2)$-primary ideal of R_S.*

Proof. Let $(x/s_1)(y/s_2) \in I_S$ for some $x, y \in R$; $s_1, s_2 \in S$. Then there are $a \in I$ and $s \in S$ with $(x/s_1)(y/s_2) = a/s$. Thus we have $sxy \in I$. Then $(sx)^2 \in I$ or $y^2 \in \delta(I)$. Hence $(s^2/s^2)(x^2/s_1^2) \in I_S$ or $y^2/s_2^2 \in \delta(I)_S$. We have $(x/s_1)^2 \in I_S$ or $(y/s_2)^2 \in (\delta(I))_S = \delta_S(I_S)$. Consequently, I_S is a $(\delta_S, 2)$-primary ideal of R_S. □

Theorem 5. *Let δ be an expansion of ideals of R. Then the following statements are equivalent:*
(1) *Every proper principal ideal is a $(\delta, 2)$-primary ideal of R.*
(2) *Every proper ideal is a $(\delta, 2)$-primary ideal of R.*

Proof. Suppose that (1) holds. Let I be a proper ideal of R and $a, b \in R$ with $ab \in I$. Then $ab \in (ab)$ and since (ab) is a $(\delta, 2)$-primary ideal of R by our assumption, we have either $a^2 \in (ab) \subseteq I$ or $b^2 \in \delta(ab) \subseteq \delta(I)$. Thus I is a $(\delta, 2)$-primary ideal of R. The converse part is obvious. □

Recall that a commutative ring R is called a von Neumann regular ring if for every $a \in R$ there exists $x \in R$ such that $a = axa$. Note that a ring R is von Neumann regular if and only if for any ideal I of R, $\sqrt{I} = I$. A commutative ring R is called Boolean if $a^2 = a$ for each $a \in R$. It is clear that every Boolean ring is von Neumann ring. In the following theorems, we characterize von Neumann regular rings in terms of 2-prime and $(\delta,2)$-primary ideals.

Theorem 6. *A ring R is von Neumann regular if and only if every 2-prime ideal of R is a prime ideal.*

Proof. ($\Rightarrow$): Let $x, y \in R$ with $xy \in I$ and $x \notin I$. Since R is a von Neumann regular ring, we have $a \in R$ with $x = ax^2$. Indeed, if $x^2 \in I$, then $ax^2 = x \in I$, a contradiction. Thus, $x^2 \notin I$. By assumption, we get $y^2 \in I$. Therefore, $y \in \sqrt{I} = I$ as R is a von Neumann regular ring.

($\Leftarrow$): If a proper ideal I of a ring R is a 2-prime, then $\sqrt{I}$ is prime in [6], Proposition 1.3, statement (1). Thus, we have $\sqrt{I} = I$ for all ideals I of R. Therefore R is von Neumann regular. □

Theorem 7. *Let R be a von Neumann regular ring and δ an expansion function of $\mathcal{I}(\mathcal{R})$. Then every $(\delta,2)$-primary ideal of R is a δ-primary ideal.*

Proof. Suppose that I is a $(\delta,2)$-primary ideal, $xy \in I$ and $x \notin I$ for some $x, y \in R$. Then there is $a \in R$ with $x = ax^2$ as R is assumed to be von Neumann regular. If $x^2 \in I$, then $ax^2 = x \in I$, a contradiction. Thus $x^2 \notin I$. Since I is $(\delta,2)$-primary, we get $y^2 \in \delta(I)$. Therefore, $y \in \delta(I)$ as R is a von Neumann regular ring. Thus I is a δ-primary ideal of R. □

Note that Theorem 6 and Theorem 7 hold for Boolean rings. An integral domain R is said to be a valuation ring if for every element a of its field of fractions K, at least one of a or a^{-1} belongs to R.

Theorem 8. *Let R be a valuation ring with the quotient field K and let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. For a proper ideal I of R, the following statements hold:*

(1) I is a $(\delta,2)$-primary ideal of R if and only if for every $a, b \in K$ with $ab \in I$ and $a^2 \notin I$, we have $b^2 \in \delta(I)$.
(2) I is a 2-prime ideal of R if and only if for every $a, b \in K$ with $ab \in I$ and $a^2 \notin I$, we have $b^2 \in I$.

Proof. (1) Suppose that I is a $(\delta,2)$-primary ideal of R and $a, b \in K$ are such that $ab \in I$ with $a^2 \notin I$. If $a \notin R$, then $a^{-1} \in R$ as R is assumed to be a valuation. Hence $b = a^{-1}ab \in I$, and so $b^2 \in I \subseteq \delta(I)$. Now assume that $a \in R$. If b is also an element of R, then the result is clear since I is a $(\delta,2)$-primary ideal of R. So assume $b \notin R$. Then $b^{-1} \in R$ and we conclude $a = abb^{-1} \in I$ which contradicts $a^2 \notin I$. Thus we are done. The converse part is obvious.

(2) It is similar to (1). □

Let $R_1, R_2, \ldots, R_n$ be commutative rings with nonzero identity, let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i \in \{1, 2, \ldots, n\}$ and $R = R_1 \times \ldots \times R_n$. For a proper ideal $I_1 \times \ldots \times I_n$, the function $\delta_\times$ defined by $\delta_\times(I_1 \times I_2 \times \ldots \times I_n) = \delta_1(I_1) \times \delta_2(I_2) \times \ldots \times \delta_n(I_n)$ is an expansion function of $\mathcal{I}(\mathcal{R})$. In the next two theorems, we characterize $(\delta, 2)$-primary ideals of $R_1 \times \ldots \times R_n$.

Theorem 9. *Let R_1 and R_2 be commutative rings with $1 \neq 0$ and $R = R_1 \times R_2$, and let δ_1, δ_2 be expansion functions of $\mathcal{I}(\mathcal{R}_1)$ and $\mathcal{I}(\mathcal{R}_2)$, respectively. Then the following statements are equivalent:*

(1) *I is a $(\delta_\times, 2)$-primary ideal of R.*

(2) *Either $I = I_1 \times R_2$, where I_1 is a $(\delta_1, 2)$-primary ideal of R_1 or $I = R_1 \times I_2$, where I_2 is a $(\delta_2, 2)$-primary ideal of R_2 or $I = I_1 \times I_2$, where I_1 and I_2 are proper ideals of R_1, R_2, respectively with $\delta_1(I_1) = R_1$ and $\delta_2(I_2) = R_2$.*

Proof. (1) ⇒ (2): Let I be a $(\delta_\times, 2)$-primary ideal of R. We know that an ideal I of R is of the form $I = I_1 \times I_2$ where I_1 and I_2 are ideals of R_1 and R_2, respectively. Without loss of generality, we may assume that $I = I_1 \times R_2$ for some proper ideal I_1 of R_1. We show that I_1 is a $(\delta_1, 2)$-primary ideal of R_1. Assume not. Then there are $a, b \in R_1$ such that $ab \in I_1$, $a^2 \notin I_1$ and $b^2 \notin \delta_1(I_1)$. We get $(a, 1)(b, 1) \in I_1 \times R_2$. It implies $(a^2, 1) \in I_1 \times R_2$ or $(b^2, 1) \in \delta_\times(I_1 \times R_2)$. Thus $a^2 \in I_1$ or $b^2 \in \delta(I_1)$, yielding a contradiction. Now suppose that both I_1 and I_2 are proper. Since $(1, 0)(0, 1) \in I_1 \times I_2$ and $(1, 0)^2, (0, 1)^2 \notin I_1 \times I_2$, we have $(1, 0)^2, (0, 1)^2 \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$. This yields $\delta_1(I_1) = R_1$ and $\delta_2(I_2) = R_2$.

(2) ⇒ (1): This side is clear. ☐

Theorem 10. *Let $R_1, R_2, \ldots, R_n$ be commutative rings with nonzero identity and $R = R_1 \times \ldots \times R_n$, where $n \geqslant 2$. Let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i = 1, \ldots, n$. Then the following statements are equivalent:*

(1) *I is a $(\delta_\times, 2)$-primary ideal of R.*

(2) *$I = I_1 \times \ldots \times I_n$ and either for some $k \in \{1, \ldots, n\}$ such that I_k is a $(\delta_k, 2)$-primary ideal of R_k and $I_j = R_j$ for all $j \in \{1, \ldots, n\} \setminus \{k\}$ or I_{α_i}'s are proper ideals of R_{α_i} for $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq \{1, 2, \ldots, n\}$ and $|\{\alpha_1, \alpha_2, \ldots, \alpha_k\}| \geqslant 2$ with $\delta_{\alpha_i}(I_{\alpha_i}) = R_{\alpha_i}$ and $I_j = R_j$ for all $j \in \{1, \ldots, n\} \setminus \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.*

Proof. It can be obtained by using mathematical induction on n. ☐

Let R be a commutative ring and M an R-module. The idealization $R(+)M = \{(r, m) \colon r \in R, m \in M\}$ is a commutative ring with addition and multiplication, respectively: $(r, m)(s, m') = (r + s, m + m')$ and $(r, m)(s, m') = (rs, rm' + sm)$ for each $r, s \in R$, $m, m' \in M$. Additionally, $I(+)N$ is an ideal of $R(+)M$, where I is an ideal of R and N is a submodule of M if and only if $IM \subseteq N$ (see [2] and [9]). In this circumstances, $I(+)N$ is called a homogeneous ideal of $R(+)M$. Recall that the radical of a homogeneous ideal is $\sqrt{I(+)N} = \sqrt{I}(+)M$, see [2]. Let δ be an expansion function of R. Clearly, $\delta_{(+)}$ is defined as $\delta_{(+)}(I(+)N) = \delta(I)(+)M$ for every ideal $I(+)N$ of $R(+)M$ is an expansion function of $R(+)M$.

Theorem 11. *Let δ be an expansion function of R and let $I(+)N$ be a homogenous ideal of $R(+)M$. Then the following statements hold:*

1. *If I is a $(\delta, 2)$-primary ideal of R and $\sqrt{I}M \subseteq N$, then $I(+)N$ is a $(\delta_{(+)}, 2)$-primary ideal of $R(+)M$.*
2. *If $I(+)N$ is a $(\delta_{(+)}, 2)$-primary ideal of $R(+)M$, then I is a $(\delta, 2)$-primary ideal of R.*

Proof. (1) Let $(r, m)(r', m') = (rr', rm' + r'm) \in I(+)N$ for some (r, m), $(r', m') \in R(+)M$. Then $rr' \in I$, and so $r^2 \in I$ or $r'^2 \in \delta(I)$. Assume that $r^2 \in I$. Then $r \in \sqrt{I}$ and so $2rm \in N$ as $\sqrt{I}M \subseteq N$. Then $(r, m)^2 = (r^2, 2rm) \in I(+)N$. Let $r'^2 \in \delta(I)$. Then $(r', m'^2 = (r'^2, 2r'm') \in \delta(I)(+)M = \delta_{(+)}(I(+)N)$.

(2) Let $rr' \in I$ for some $r, r' \in R$. Then $(r, 0)(r', 0) \in I(+)N$. Hence $(r, 0)^2 = (r^2, 0) \in I(+)N$ or $(r', 0)^2 = (r'^2, 0) \in \delta_{(+)}(I(+)N)$. Therefore, we have $r^2 \in I$ or $r'^2 \in \delta(I)$, as needed. □

Corollary 4. *Let $I(+)N$ be a homogeneous ideal of $R(+)M$ and $(N : M) = \sqrt{(N : M)}$. Then I is a $(\delta, 2)$-primary ideal of R if and only if $I(+)N$ is a $(\delta_{(+)}, 2)$-primary ideal of $R(+)M$.*

More general than the $(\delta, 2)$-primary ideal of a commutative ring, the concept of the (δ, n)-primary ideal of R, where n is a positive integer can be defined. We give just the definition of this concept which may be inspiring for other work:

Definition 2. Let R be a commutative ring with nonzero identity, δ an expansion function of $\mathcal{I}(\mathcal{R})$ and n a positive integer.We call a proper ideal I of R a (δ, n)-primary ideal if whenever $a, b \in R$ with $ab \in I$, then either $a^n \in I$ or $b^n \in \delta(I)$. In particular, for $n = 1, 2$, it is a δ-primary and a $(\delta, 2)$-primary ideal, respectively.

3. EXAMPLES

Example 1. Let R be a valuation ring. Then every proper ideal is $(\delta, 2)$-primary by [6], Theorem 2.4.

By Proposition 1, statement (4), we obtain Figure 2. But the converse of the relation in Figure 2 is not satisfied in general (see the next example).

$$\boxed{\text{prime ideal}} \quad\Longrightarrow\quad \boxed{(\delta_1, 2)\text{-primary ideal}}$$

Figure 2. Relation between primary ideal and $(\delta_1, 2)$-primary ideal

Example 2. Let R be a subring of $\mathbb{Z}[X]$ which consists of polynomials such that the coefficients of X can be divided by 3. Consider the ideal $Q = (9X^2, 3X^3, X^4, X^5, X^6)$ of R. One can see that Q is a $(\delta_1, 2)$-primary ideal of R, where $\delta_1(Q) = (3X, X^2, X^3)$ is a prime ideal of R. However Q is not a primary ideal of R since $3X^3 \in Q$ and $X^3 \notin Q$ but $3^n \notin \sqrt{Q} = (3X, X^2, X^3)$ for all positive integers n.

The following example shows that the intersection of two $(\delta, 2)$-primary ideals of a commutative ring need not be $(\delta, 2)$-primary in general:

Example 3. Consider the ring $R = \mathbb{Z}_{12}$ and the ideals $I = 4\mathbb{Z}_{12}$ and $J = 3\mathbb{Z}_{12}$ of R. Then clearly both I and J are $(\delta_i, 2)$-primary for $i = 0, 1$. However, $I \cap J = (0)$ is not: $3 \cdot 4 \in (0)$ but neither $3 \in (0)$ nor $4 \in \delta_i((0))$ for $i = 0, 1$.

The next examples demonstrate that the converses of the relations between the $(\delta, 2)$-primary ideal and other classical ideals in Figure 1 do not hold in general. The following example shows that the converse of Proposition 1 (5) is not satisfied in general.

Example 4. Consider the ring $R = F[X, Y]$ where F is a field. Let $I = (X^3, XY, Y^3)$. Then the radical of I, $(X, Y) \in \mathrm{Max}(R)$, is the set of all maximal ideals of R. It is clear that I is a $(\delta_1, 2)$-primary ideal. But it is not a 2-prime ideal.

The following two examples show that the converse of Proposition 1, statement (6) is not always true.

Example 5. Consider the ring $\mathbb{Z}_8$ and let $\delta\colon \mathbb{Z}_8 \to \mathbb{Z}_8$ be an expansion of ideals of $\mathbb{Z}_8$ defined by $\delta(J) = J + (4)$ for all ideals J of $\mathbb{Z}_8$. Then the zero ideal is a $(\delta, 2)$-primary ideal of $\mathbb{Z}_8$, but it is neither prime nor δ-primary. Indeed, (0) is not a δ-primary ideal of $\mathbb{Z}_8$ as $4 \cdot 2 \in (0)$ but $4 \notin (0)$, $2 \notin \delta((0)) = (4)$.

Example 6. A proper ideal (4) of $\mathbb{Z}$ is a $(\delta_0, 2)$-primary ideal but it is not a δ_0-primary ideal of $\mathbb{Z}$.

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