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Article

Classical 1-Absorbing Primary Submodules

Zeynep Yılmaz Uçar [1], Bayram Ali Ersoy [1], Ünsal Tekir [2], Ece Yetkin Çelikel [3,*] and Serkan Onar [4]

1 Department of Mathematics, Yildiz Technical University, Istanbul 34220, Türkiye; zeynep.ucar@std.yildiz.edu.tr (Z.Y.U.); ersoya@yildiz.edu.tr (B.A.E.)
2 Department of Mathematics, Marmara University, Istanbul 34722, Türkiye; utekir@marmara.edu.tr
3 Department of Basic Sciences, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep 27010, Türkiye
4 Department of Mathematical Engineering, Yildiz Technical University, Istanbul 34220, Türkiye; sonar@yildiz.edu.tr
* Correspondence: yetkinece@gmail.com or ece.celikel@hku.edu.tr

Abstract: Over the years, prime submodules and their generalizations have played a pivotal role in commutative algebra, garnering considerable attention from numerous researchers and scholars in the field. This papers presents a generalization of 1-absorbing primary ideals, namely the classical 1-absorbing primary submodules. Let $\Re$ be a commutative ring and M an $\Re$-module. A proper submodule K of M is called a classical 1-absorbing primary submodule of M, if $xyz\eta \in K$ for some $\eta \in M$ and nonunits $x, y, z \in \Re$, then $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geq 1$. In addition to providing various characterizations of classical 1-absorbing primary submodules, we examine relationships between classical 1-absorbing primary submodules and 1-absorbing primary submodules. We also explore the properties of classical 1-absorbing primary submodules under homomorphism in factor modules, the localization modules and Cartesian product of modules. Finally, we investigate this class of submodules in amalgamated duplication of modules.

Keywords: primary submodule; 1-absorbing primary submodule; classical primary submodule; classical 1-absorbing primary submodule

MSC: 13A15; 13C05; 13C13

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1. Introduction

In this paper, all rings are considered to be commutative and possess a nonzero identity element. All modules are assumed to be unitary. It has been well known for years that a proper ideal, J (resp. Q), of a ring, $\Re$, is defined as prime (resp. primary) if whenever $xy \in J$ (resp. $xy \in Q$) for some $x, y \in R$, then either $x \in J$ or $y \in J$ (resp. $x \in Q$ or $y \in \sqrt{Q}$). The notion of prime ideals has been extended to prime submodules by several authors, see for example [1–3]. Let M be an $\Re$-module. Recall that a prime (resp. primary) submodule is a proper submodule K of M with the property that, for $x \in \Re$ and $\eta \in M$, $x\eta \in K$ implies that $\eta \in K$ or $x \in (K :_{\Re} M)$. There are several ways to generalize the concept of prime submodules. In 2004, Behboodi and Koohy introduced the notion of classical prime submodules in [4]. A proper submodule K of M is called a classical prime submodule, if, for each $\eta \in M$ and $x, y \in \Re$, $xy\eta \in K$ implies that $x\eta \in K$ or $y\eta \in K$ [5,6]. It is important to observe that every prime submodule is also a classical prime, but the converse does not hold in general. Thus far, the notion of classical prime submodules has garnered significant attention from many researchers and has been extensively examined in various academic papers. For instance, Baziar and Behboodi [7] prescribed a classical primary submodule as follows: a proper submodule K of M is said to be a classical primary submodule if $xy\eta \in K$ for some $x, y \in \Re$, and $\eta \in M$ implies that $x\eta \in K$ or $y^t\eta \in K$ for some $t \geq 1$. For more information regarding classical prime submodules, the researcher is directed to [7–11]. On the other hand, in [12], Badawi and Yetkin Çelikel gave an extension of primary ideals, namely 1-absorbing primary (briefly 1-a.p) ideals. A proper ideal, J, of a ring, $\Re$, is called a 1-a.p ideal of $\Re$ if $xyz \in J$ for some nonunits $x, y, z \in \Re$, then $xy \in J$ or $z \in \sqrt{J}$. Afterwards, Yetkin Çelikel [13] extended this notion to modules. A proper submodule K of M is said to be a 1-a.p submodule of M if $xy\eta \in K$ for some nonunits $x, y \in \Re$ and $\eta \in M$, then either $xy \in (K :_{\Re} M)$ or $\eta \in M - rad(K)$, where $M - rad(K)$ is the prime radical of K.

In the interest of completeness, we provide some definitions that will be required throughout this study. Given a ring, $\Re$, assume J as one of its ideals. By $\sqrt{J} = \{a \in \Re \mid a^k \in J \text{ for some } k \in \mathbb{N}\}$, is defined as the radical of J, meaning it is the intersection of all prime ideals that contain J. Consider M as an $\Re$-module and K as a submodule of M. We will represent by $(K :_{\Re} M)$ the residual of K by M, which is the set of all $x \in \Re$, such that $xM \subseteq K$. An $\Re$-module is said to be faithful provided that $(0 :_{\Re} M) = 0$ [14]. An $\Re$-module M is referred to as a multiplication module if each submodule K of M can be written as JM for some ideal J of $\Re$ [15]. Let K and L be submodules of a multiplication $\Re$-module M with $K = J_1M$ and $L = J_2M$ for some presentation ideals, J_1 and J_2, of $\Re$. Ameri in his paper [16], defined the product of submodules of multiplication modules as follows: the product of K and L denoted by KL is defined by $KL := J_1J_2M$ [16] (Theorem 3.4). The M-radical of K, expressed by $M - rad(K)$, is specified as the intersection of all prime submodules of M that contain K. In the case that there is no such a prime submodule, then $M - rad(K) = M$. If M is a multiplication $\Re$-module, then $M - rad(K)$=$\{\eta \in M : \eta^t \subseteq K \text{ for some } t \geq 0\}$ [16] (Theorem 3.13). Motivated from the studies mentioned above, in this study, we introduce and study the concept of classical 1-absorbing primary (briefly, classical 1-a.p) submodules. A proper submodule K of M is called classical 1-a.p submodule if whenever nonunits $x, y, z \in \Re$, $\eta \in M$ and $xyz\eta \in K$ implies that either $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geq 1$. Among the results presented in Section 2, we first demonstrate that every classical primary submodule is a classical 1-a.p submodule, but the converse is not true in general (see Example 1). Various characterizations for classical 1-a.p submodules are given (see Theorems 1–5). Also, we examine the behavior of classical 1-a.p submodules under homomorphisms, in factor modules, in localization of modules, in Cartesian product of modules and in multiplication modules (see Propositions 2–6). Additionally, we investigate the classical 1-a.p submodule of tensor product $\Im \otimes M$ for a flat $\Re$-module $\Im$ and any $\Re$-module M (see Proposition 4). It is widely recognized in the union of finitely many prime ideals, it must be contained in at least one of these prime ideals. However, this condition will generally not hold for arbitrary ideals. For instance, consider $\Re = \mathbb{Z}_2[\alpha, \beta, \gamma]/(\alpha, \beta, \gamma)^2$ and $J = \{\overline{0}, \overline{\alpha}, \overline{\beta}, \overline{\alpha + \beta}\}$. Let $I = \{\overline{0}, \overline{\alpha}\}$, $P = \{\overline{0}, \overline{\beta}\}$ and $Q = \{\overline{0}, \overline{\alpha + \beta}\}$. Then $J \subseteq I \cup P \cup Q$ and J is not contained in one of the ideals I, P, Q of $\Re$. In the context of rings where this condition holds, u-rings and um-rings have been introduced, as mentioned in [17]. This study states that, if an ideal is contained in the union of finitely many ideals, this implies that it is contained in at least one of these ideals, and such rings are called u-rings. Additionally, if a submodule is contained in the union of finitely many submodules, this implies that it is equal to one of these submodules, and such rings are called um-rings. In our work, we have characterized classical 1-a.p submodules in ring structures that satisfy these properties (see Theorems 6 and 7).

Section 3 is devoted to the investigate of classical primary and classical 1-a.p submodules within the context of the amalgamated duplication (briefly a.d) of modules over commutative rings. Consider $\Re$ to be a ring and J to be an ideal of $\Re$. The amalgamated duplication of a ring, $\Re$, along an ideal J, denoted by $\Re \bowtie J$, was firstly introduced and studied by D'anna and Fontana in [18]. The amalgamated duplication $\Re \bowtie J = \{(x, x + j) : x \in \Re, j \in J\}$ of a ring, $\Re$, along an ideal J is a special subring of $\Re \times \Re$, with addition and multiplication performed component by component. In fact, $\Re \bowtie J$ is a commutative subring having the same identity of $\Re \times \Re$. A more comprehensive generalization of amalgamation rings was conducted by D'anna, Finocchiaro and Fontana in [19]. The concept of ring amalgamation holds a significant position in commutative algebra and has been extensively explored by numerous renowned algebraists. Then, the idea of a.d of a ring was extended to the context of modules by Bouba, Mahdou and Tamekkante, as described below. Let M be an $\Re$-module and J be an ideal of $\Re$. The amalgamated duplication of an $\Re$-module M along an ideal J, denoted by $M \bowtie J = \{(\eta, \eta + \eta') : \eta \in M,\ \eta' \in JM\}$ is an $\Re \bowtie J$-module with componentwise addition and the following scalar multiplication: $(x, x+j)(\eta, \eta+\eta') = (x\eta, (x+j)(\eta+\eta'))$ for each $(x, x+j) \in \Re \bowtie J$ and $(\eta, \eta+\eta') \in M \bowtie J$ [20]. Note that if we consider $M = \Re$ as an $\Re$-module, then the a.d $M \bowtie J$ of the $\Re$-module M along the ideal J and the a.d $\Re \bowtie J$ of the ring $\Re$ along the ideal J coincide. If K is a submodule of M, then it can be easily verified that $K \bowtie J = \{(\eta, \eta+\eta') : \eta \in K, \eta' \in JM\}$ is an $\Re \bowtie J$-submodule of $M \bowtie J$. Now, one can naturally ask the classical primary and classical 1-a.p submodules of $M \bowtie J$. In this section, first we find a useful equality for the residual of $K \bowtie J$ by $(x, x+j) \in \Re \bowtie J$ or $(\eta, \eta+\eta') \in M \bowtie J$ (see Lemma 2). Then by Lemma 2 and Theorem 4, we determine the classical primary and classical 1-a.p submodules of $M \bowtie J$ (see Theorem 9).

In conclusion, we have demonstrated that many of the results obtained with 1-a.p ideals in commutative algebra are achieved through the new structure we established in module theory. By presenting an example of a module that is a classical 1-a.p submodule but not a classical primary submodule, we have shown that the new framework we developed encompasses a broader class. Furthermore, we have conducted an in-depth analysis of the similarities between the new structure and various existing generalizations of prime submodules.

2. Characterizations of Classical 1-Absorbing Primary Submodules

Within this part, we will give characterizations of classical 1-a.p submodules of R-module M. We start with our main definition.

Definition 1. *Let $\Re$ denote a commutative ring and M represent an $\Re$-module. A proper submodule K of M is said to be a classical 1-a.p submodule if $xyz\eta \in K$ for some elements $x, y, z \in \Re$ that are not units and $\eta \in M$, then either $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geqslant 1$.*

Note that 1-a.p ideals of a ring, $\Re$, and classical 1-a.p submodules of $\Re$-module $\Re$ coincide. It follows from the definition that every classical primary submodule is a classical 1-a.p submodule; however, the inverse of this implication is not valid in general.

Example 1. *Let $\Re = \Im[a, b]$, where $\Im$ is a field, $J = \langle a, b \rangle$, the localization ring $\breve{\Re} = \Re_J$ and $M = \breve{\Re}$. Consider the $\breve{\Re}$-submodule of M. Then, $K = bJ_J = \langle ab, b^2 \rangle$ is a classical 1-a.p submodule of of $\breve{\Re}$-module M by [12] (Theorem 5). However, it is not classical primary as $ba1 \in K$ but neither $b1 \in K$ or $a^t1 \in K$.*

We provide a characterization of classical 1-a.p submodules of an $\Re$-module as follows.

Theorem 1. *Consider M as an $\Re$-module and K as a proper submodule of M. Then the subsequent conditions are equivalent.*

1. *K is a classical 1-a.p submodule of M;*
2. *For every $\eta \in M \setminus K$, $(K :_\Re \eta)$ is a 1-a.p ideal of $\Re$;*
3. *For every $\eta \in M \setminus K$, $(K :_\Re \eta)$ is a 1-a.p ideal of $\Re$ and $\left\{ \sqrt{(K :_\Re \eta)} : \eta \in M \setminus K \right\}$ is a chain of prime ideals of $\Re$.*

Proof. (1) $\Rightarrow$ (2) Assume that K is a classical 1-a.p submodule of M. Let $\eta \in M \setminus K$ and nonunit elements $x, y, z \in \Re$ with $xyz \in (K :_\Re \eta)$. Then $xyz\eta \in K$, which implies either $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geq 1$. Hence, $xy \in (K :_\Re \eta)$ or $z^t \in (K :_\Re \eta)$ for some $t \geq 1$. Thus, $xy \in (K :_\Re \eta)$ or $z \in \sqrt{(K :_\Re \eta)}$. Consequently, $(K :_\Re \eta)$ is a 1-a.p ideal of $\Re$.

(2) $\Rightarrow$ (3) For each $\eta_1, \eta_2 \in M \setminus K$, we have $\sqrt{(K :_\Re \eta_1)} \cap \sqrt{(K :_\Re \eta_2)} \subseteq \sqrt{(K :_\Re \eta_1 + \eta_2)}$. If $\sqrt{(K :_\Re \eta_1 + \eta_2)}$ is proper, it is prime by [13] (Proposition 1), depending on whether $\eta_1 + \eta_2$ belongs to K or not, and we conclude that either $\sqrt{(K :_\Re \eta_1)} \subseteq \sqrt{(K :_\Re \eta_1 + \eta_2)}$

or $\sqrt{(K :_{\Re} \eta_2)} \subseteq \sqrt{(K :_{\Re} \eta_1 + \eta_2)}$. It follows that $\sqrt{(K :_{\Re} \eta_1)} \subseteq \sqrt{(K :_{\Re} \eta_2)}$ or $\sqrt{(K :_{\Re} \eta_2)} \subseteq \sqrt{(K :_{\Re} \eta_1)}$. Hence, $\left\{ \sqrt{(K :_{\Re} \eta)} : \eta \in M \setminus K \right\}$ is a chain of prime ideals of $\Re$.

(3) $\Rightarrow$ (1) Suppose that $(K :_{\Re} \eta)$ is a 1-a.p ideal of R for every $\eta \in M \setminus K$. Let $xyz\eta \in K$ for some nonunit elements $x, y, z \in \Re$ and $\eta \in M$. If $\eta \in K$, then we are done. So, assume that $\eta \notin K$, then $xyz \in (K :_{\Re} \eta)$. Since $(K :_{\Re} \eta)$ is a 1-a.p ideal, we have either $xy \in (K :_{\Re} \eta)$ or $z^t \in (K :_{\Re} \eta)$ for some $t \geq 1$. Therefore, $xy\eta \in K$ or $z^t\eta \in K$. Thus, K is a classical 1-a.p submodule of M. □

Let M be an $\Re$-module. It is known from [13] (Proposition 1) that, for every 1-a.p submodule K of M, $\sqrt{(K :_{\Re} M)}$ is a prime ideal of $\Re$. Here, we show that this fact also true for a classical 1-a.p submodule K of a finitely generated module of M, and, in this case, we say that K is a P-classical 1-a.p submodule where $P = \sqrt{(K :_{\Re} M)}$.

Lemma 1. *Consider M as a finitely generated $\Re$-module and K as a classical 1-a.p submodule of M. Then $(K :_{\Re} M)$ is a 1-a.p ideal and $\sqrt{(K :_{\Re} M)}$ is a prime ideal of $\Re$. Furthermore, if M is a multiplication $\Re$-module, then M-$rad(K)$ is a prime submodule of M.*

Proof. Let $x, y, z \in \Re$ be nonunit elements, such that $xyz \in (K :_{\Re} M)$. Since K is a classical 1-a.p submodule, for each $\eta \in M$, either $xy\eta \in K$ or $z^t\eta \in K$ for some $t \in \mathbb{N}$. Now let $X := \{\eta \in M \mid xy\eta \in K\}$ and $Y := \{\eta \in M \mid z^t\eta \in K \text{ for some } t \in \mathbb{N}\}$. Then one can easily see that X and Y are submodules of M and $M = X \cup Y$. It follows that $M = X$ or $M = Y$. If $M = X$, then $xyM \subseteq K$ and so $xy \in (K :_{\Re} M)$. Now let $M = Y$ and, since M is finitely generated, $z^tM \subseteq K$ for some $t \in \mathbb{N}$. Thus, $z^t \in (K :_{\Re} M)$ and therefore $(K :_{\Re} M)$ is a 1-a.p of R. □

The subsequent example demonstrates that if $\sqrt{(K :_{\Re} M)}$ is a prime ideal of $\Re$, then K is not necessarily a classical 1-a.p submodule of M in general.

Example 2. *Let $\Re = \mathbb{Z}$ and $M = \mathbb{Z}_a \oplus \mathbb{Z}_b \oplus \mathbb{Q}$, where a, b are two distinct prime integers. Consider the zero submodule $K = \{0\}$ of M. Then $\sqrt{(K :_{\Re} M)} = \{0\}$ is a prime ideal of $\Re$. However, K is not a classical 1-a.p submodule since $a \cdot b \cdot c \cdot (\bar{1}, \bar{1}, 0) = (\bar{0}, \bar{0}, 0)$, $a \cdot b \cdot (\bar{1}, \bar{1}, 0) \notin K$ and $c^t \cdot (\bar{1}, \bar{1}, 0) \notin K$ for all $t \geq 1$, where c is a prime number different from a and b.*

Now, we present some equivalent conditions to characterize classical 1-a.p submodules in finitely generated multiplication modules.

Theorem 2. *Let M be a finitely generated multiplication $\Re$-module. For a proper submodule K of M, the following assertions are equivalent.*

1. *K is a 1-a.p submodule;*
2. *K is a classical 1-a.p submodule;*
3. *$(K :_{\Re} M)$ is a 1-a.p ideal of $\Re$.*

Proof. (1) $\Rightarrow$ (2) We show that any 1-a.p submodule of a multiplication module is a classical 1-a.p submodule. Indeed, suppose that K is a 1-a.p submodule of a multiplication $\Re$-module M and $xyz\eta \in K$ for some nonunits $x, y, z \in \Re$ and $\eta \in M$. Then $xy \in (K :_{\Re} M)$ or $z\eta \in M - rad(K) = \sqrt{(K :_{\Re} M)}M$, as M is a multiplication module. Then, $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geq 1$, and thus K is a classical 1-a.p submodule of M.

(2) $\Rightarrow$ (3) Lemma 1.

(3) $\Rightarrow$ (1) Let $xy\eta \in K$, where nonunit elements $x, y \in \Re$ and $\eta \in M \setminus M\text{-}rad(K)$. Since M is a multiplication module, $\Re\eta = JM$ for some ideal J of $\Re$. Thus $xyJM \subseteq K$ and $xyJ \subseteq (K :_{\Re} M)$. Since $(K :_{\Re} M)$ is a 1-a.p ideal and $J \nsubseteq \sqrt{(K :_{\Re} M)}$, we obtain $xy \in (K :_{\Re} M)$. Therefore, K is a 1-a.p submodule of M. □

We call an $\Re$-module M 1-a.p compatible if its classical primary and classical 1-a.p submodules are the same. A ring, $\Re$, is said to be 1-a.p compatible if every finitely generated

$\Re$-module is 1-a.p compatible. Let $\Re$ be a ring. Next, in an $\Re$-module M, we show that, if $dim(R) = 0$, then $\Re$ is a 1-a.p compatible.

Theorem 3. *Every zero dimensional ring (in particular, a field or an Artinian local ring) is 1-a.p compatible.*

Proof. Assume that $\Re$ is zero dimensional but $\Re$ is not 1-a.p compatible. Then there exists a classical 1-a.p submodule K of M that is not classical primary. Hence, for $\eta \in M - K$, $(K :_\Re \eta)$ is a 1-a.p ideal but not a primary ideal. Then $\Re$ is a local ring that has only maximal ideal q [12] (Theorem 3). But, by the assumption that every prime ideal of $\Re$ is maximal, it follows that $\Re$ has only one prime ideal, and this ideal is q. Nevertheless, $\sqrt{(K :_\Re \eta)}$ qualifies as a prime ideal given that $(K :_\Re \eta)$ is a 1-a.p ideal, as substantiated by [12] (Theorem 2). It follows that $\sqrt{(K :_\Re \eta)}$ is a maximal ideal. Thus, we conclude $(K :_\Re \eta)$ is a primary ideal, which is a contradiction. Hence, $\Re$ is 1-a.p compatible. □

Consider M as an $\Re$-module and K as a submodule of M. For every $x \in \Re$, $\{\eta \in M \mid x\eta \in K\}$ is denoted by $(K :_M x)$. It is evident that $(K :_M x)$ is a submodule of M that encompasses K. In the following theorem, we provide a characterization of a classical 1-a.p submodule.

Theorem 4. *Consider M as an $\Re$-module and K as a proper submodule of M. The following statements hold with the same equivalence:*

1. *K is a classical 1-a.p submodule of M;*
2. *For each nonunits $x, y, z \in \Re$, $(K :_M xyz) \subseteq (K :_M xy) \cup \left(\cup_{t\geq 1}(K :_M z^t)\right)$;*
3. *For each nonunits $x, y \in \Re$ and $\eta \in M$ with $xy\eta \notin K$, $(K :_\Re xy\eta) \subseteq \sqrt{(K :_\Re \eta)}$;*
4. *For each nonunits $x, y \in \Re$ and every ideal J of $\Re$ and $\eta \in M$ with $xyJ\eta \subseteq K$, either $xy\eta \in K$ or $J \subseteq \sqrt{(K :_\Re \eta)}$;*
5. *For each ideals I, J, L of $\Re$ and $\eta \in M$ with $IJL\eta \subseteq K$, either $IJ\eta \subseteq K$ or $L \subseteq \sqrt{(K :_\Re \eta)}$.*

Proof. (1) $\Rightarrow$ (2) Assume that K is a classical 1-a.p submodule and choose nonunits $x, y, z \in \Re$. Let $\eta \in (K :_M xyz)$. Then we have $xyz\eta \in K$, which implies that $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geq 1$. Therefore, we deduce that $\eta \in (K :_M xy) \cup \left(\cup_{t\geq 1}(K :_M z^t)\right)$, that is, $(K :_M xyz) \subseteq (K :_M xy) \cup \left(\cup_{t\geq 1}(K :_M z^t)\right)$.

(2) $\Rightarrow$ (3) Let $xy\eta \notin K$ for some nonunits $x, y \in \Re$ and $\eta \in M$. Choose $z \in (K :_\Re xy\eta)$. If z is a unit, then $(K :_\Re xy\eta) = \Re$ and $xy\eta \in K$, a contradiction. Hence z is a nonunit. Since we have $xyz\eta \in K$, so by (2) we conclude that $\eta \in (K :_M xyz) = (K :_M xy) \cup \left(\cup_{t\geq 1}(K :_M z^t)\right)$. Since $xy\eta \notin K$, we have $z^t\eta \in K$ for some $t \geq 1$, which implies that $z \in \sqrt{(K :_\Re \eta)}$. Then we obtain $(K :_\Re xy\eta) \subseteq \sqrt{(K :_\Re \eta)}$.

(3) $\Rightarrow$ (4) Suppose that $xyJ\eta \subseteq K$ for some nonunits $x, y \in \Re$, an ideal J of $\Re$ and $\eta \in M$. Hence $J \subseteq (K :_\Re xy\eta)$. If $xy\eta \in K$, then the proof is complete. So, suppose $xy\eta \notin K$ and by part (3) we obtain $J \subseteq \sqrt{(K :_\Re \eta)}$.

(4) $\Rightarrow$ (5) Let $IJL\eta \subseteq K$ for some ideals I, J, L of $\Re$ and $\eta \in M$ with $IJ\eta \nsubseteq K$. Hence there exist $x \in I$, $y \in J$, such that $xy\eta \notin K$. Since $xyL\eta \subseteq K$, (4) implies that $L \subseteq \sqrt{(K :_\Re \eta)}$, as required.

(5) $\Rightarrow$ (1) Let $xyz\eta \in K$ for some nonunit elements $x, y, z \in \Re$ and $\eta \in M$. Taking $I = (x)$, $J = (y)$ and $L = (z)$ in (5), we are done. □

If $\Re$ is a Noetherian ring, we conclude a further characterization for classical 1-a.p submodules of $\Re$-module M.

Theorem 5. *Consider M as a multiplication $\Re$-module, where $\Re$ is a Noetherian ring. For a proper submodule K of M, the subsequent assertions are equivalent.*

1. *K is a classical 1-a.p submodule of M;*
2. *If $IJL\eta \subseteq K$ for some proper submodules I, J, L of M and $\eta \in M$, then either $IJ\eta \subseteq K$ or $L^t\eta \subseteq K$ for some $t \geq 1$.*

Proof. (1) $\Rightarrow$ (2) Suppose that K is a classical 1-a.p submodule and $IJL\eta \subseteq K$ for some proper submodules I, J, L of M and $\eta \in M$. Since M is multiplication, we can write $I = VM$, $J = YM$, $L = ZM$ for some proper ideals V, Y, Z of $\Re$. Then, note that $VYZ\eta \subseteq K$. As K is a classical 1-a.p submodule, we have either $VY\eta \subseteq K$ or $Z \subseteq \sqrt{(K :_{\Re} \eta)}$ by Theorem 4. Since $\Re$ is Noetherian, we have $Z^t\eta \subseteq K$ for some $t \geq 1$. This implies that $IJ\eta \subseteq K$ or $L^t\eta \subseteq K$ for some $t \geq 1$.

(2) $\Rightarrow$ (1) Suppose that $VYZ\eta \subseteq K$ for some proper ideals V, Y, Z of $\Re$ and $\eta \in M$. Now, put $I = VM$, $J = YM$, $L = ZM$ and note that $IJL\eta \subseteq K$. Thus we have $IJ\eta \subseteq K$ or $L^t\eta \subseteq K$ for some $t \geq 1$. If $IJ\eta \subseteq K$, we have $VY(\Re\eta : M)M = VY(\Re\eta) \subseteq K$, which implies that $VY\eta \subseteq K$. If $L^t\eta \subseteq K$ for some $t \geq 1$, then similarly we have $Z^t\eta \subseteq K$ for some $t \geq 1$. Then, by Theorem 4, K is a classical 1-a.p submodule of M. □

Note that intersections of two classical 1-a.p submodules need not to be a classical 1-a.p submodule. Consider the $\mathbb{Z}-$module $M = \mathbb{Z}_6$, $N = 2\mathbb{Z}_6$ and $K = 3\mathbb{Z}_6$. Then, $N \cap K = (\overline{0})$ is not a classical 1-a.p submodule of M, as $2.2.3.1 \in (\overline{0})$, but $2.2.1 \notin (\overline{0})$ and $3^t.1 \notin (\overline{0})$ for all $t \geq 1$. On the other hand, the intersection of a family of comparable classical 1-a.p submodules is a classical 1-a.p submodule.

Remark 1. *Let M be an $\Re$-module and $\{K_i : i \in \Delta\}$ be a descending chain of a classical 1-a.p submodule of M. Then $\cap_{i\in\Delta}K_i$ $(i \in 1, 2, \ldots, n)$ is a classical 1-a.p submodule of M.*

Proof. Let $xyz\eta \in \cap_{i\in\Delta}K_i$ for some nonunits $x, y, z \in R$ and $\eta \in M$. Then we have $xyz\eta \in K_i$ for each $i \in \Delta$. Assume that $xy\eta \notin \cap_{i\in\Delta}K_i$. Then there exists $j \in \Delta$, such that $xy\eta \notin K_j$. Since $xyz\eta \in K_j$ and K_j is a classical 1-a.p submodule, we have $z^t\eta \in K_j$ for some $t \geq 1$. Now choose $i \in \Delta$. Then either $K_j \subseteq K_i$ or $K_i \subseteq K_j$. If $K_j \subseteq K_i$, then $z^t\eta \in K_i$ for some $t \geq 1$. If $K_i \subseteq K_j$, then $xy\eta \notin K_i$ and $xyz\eta \in K_i$. Since K_i is a classical 1-a.p submodule of M, we have $z^t\eta \in K_i$ for some $t \geq 1$. Then we conclude that $z^t\eta \in \cap_{i\in\Delta}K_i$ for some $t \geq 1$. Thus, $\cap_{i\in\Delta}K_i$ is a classical 1-a.p submodule of M. □

Now, we investigate classical 1-a.p submodules over u and um-rings.

Proposition 1. *Let K be a classical 1-a.p submodule of an $\Re$-module M. Then*

1. *For all nonunits $x, y, z \in \Re$ and $\eta \in M$, $(K :_{\Re} xyz\eta) \subseteq (K :_{\Re} xy\eta) \cup \left(\cup_{t\geq1}\left(K :_{\Re} z^t\eta\right)\right)$.*
2. *If $\Re$ is a u-ring, then, for all nonunits $x, y, z \in \Re$ and $\eta \in M$, $(K :_{\Re} xyz\eta) \subseteq (K :_{\Re} xy\eta)$ or $(K :_{\Re} xyz\eta) \subseteq \left(K :_{\Re} z^t\eta\right)$ for some $t \geq 1$.*

Proof. (1) Let $r \in (K :_{\Re} xyz\eta)$. Then $xyz(r\eta) \in K$ and, since K is a classical 1-a.p submodule, we obtain either $xy(r\eta) \in K$ or $z^t(r\eta) \in K$ for some $t \geq 1$. Thus, $r \in (K :_{\Re} xy\eta)$ or $r \in \left(K :_{\Re} z^t\eta\right)$ for some $t \geq 1$.Thus, we conclude that $r \in (K :_{\Re} xy\eta) \cup \left(K :_{\Re} z^t\eta\right)$, that is, $(K :_{\Re} xyz\eta) \subseteq (K :_{\Re} xy\eta) \cup \left(\cup_{t\geq1}\left(K :_R z^t\eta\right)\right)$.

(2) Apply part (1). □

If $\Re$ is a um-ring, then we have a further characterization for classical 1-a.p submodules of $\Re$-modules.

Theorem 6. *Let $\Re$ be a um-ring, M be an $\Re$-module and K be a proper submodule of M. The following statements are equivalent.*

1. *K is a classical 1-a.p submodule of M;*
2. *For all nonunits $x, y, z \in \Re$, $(K :_M xyz) = (K :_M xy)$ or $(K :_M xyz) \subseteq \left(N :_M z^t\right)$ for some $t \geq 1$;*
3. *For all nonunits $x, y, z \in \Re$ and every submodule L of M; $xyzL \subseteq K$ implies that $xyL \subseteq K$ or $z^tL \subseteq K$ for some $t \geq 1$;*
4. *For all nonunits $x, y \in \Re$ and every submodule L of M with $xyL \nsubseteq K$, $(K :_{\Re} xyL) \subseteq \sqrt{(K :_{\Re} L)}$;*

5. *For all nonunits $x, y \in \Re$, every ideal J of $\Re$ and every submodule L of M with $xyJL \subseteq K$, this implies that $xyL \subseteq K$ or $J \subseteq \sqrt{(K :_{\Re} L)}$;*
6. *For all ideals I, J, P of $\Re$ and every submodule L of M with $IJPL \subseteq K$, this implies that $IJL \subseteq K$ or $P \subseteq \sqrt{(K :_{\Re} L)}$.*

Proof. $(1) \Rightarrow (2)$ Follows from Theorem 4 $(1) \Rightarrow (2)$.

$(2) \Rightarrow (3)$ Since $xyzL \subseteq K$, we have $L \subseteq (K : xyz)$. The rest follows from (2).

$(3) \Rightarrow (4)$ Consider $xyL \nsubseteq K$ for some nonunits $x, y \in \Re$ and some submodule L of M. Let $z \in (K :_{\Re} xyL)$.Then z is a nonunit and $xyzL \subseteq K$. By part (3), we have $z^t L \subseteq K$ for some $t \geq 1$, which implies that $z \in \sqrt{(K :_{\Re} L)}$. Hence, we obtain $(K :_{\Re} xyL) \subseteq \sqrt{(K :_{\Re} L)}$.

$(4) \Rightarrow (5)$ Suppose that $xyJL \subseteq K$ for some nonunits $x, y \in \Re$, some proper ideal J of $\Re$ and some submodule L of M. We may assume that $xyL \nsubseteq K$. Then, by part (4), we have $J \subseteq (K :_{\Re} xyL) \subseteq \sqrt{(K :_{\Re} L)}$, that is, $J \subseteq \sqrt{(K :_{\Re} L)}$.

$(5) \Rightarrow (6)$ Let $IJPL \subseteq K$ for some ideals I, J, P of $\Re$ and some submodule L of M. Suppose that $IJL \nsubseteq K$. Then $xyL \nsubseteq K$ for some $x \in I$ and $y \in J$. Since $xyPL \subseteq K$, from (5) we conclude that $P \subseteq \sqrt{(K :_{\Re} L)}$, so we are done.

$(6) \Rightarrow (1)$ Straightforward. □

We continue with a characterization of a classical 1-a.p submodule of a faithful multiplication $\Re$-module M, where $\Re$ is a Noetherian um-ring in terms of submodules of M.

Theorem 7. *Consider $\Re$ as a Noetherian um-ring, M as a multiplication $\Re$-module that is faithful and K as a proper submodule of M. The following statements are equivalent.*

1. *K is a classical 1-a.p submodule of M;*
2. *If $K_1K_2K_3K_4 \subseteq K$ for some submodules K_1, K_2, K_3 , K_4 of M, then either $K_1K_2K_4 \subseteq K$ or $K_3^tK_4 \subseteq K$ for some $t \geq 1$;*
3. *If $K_1K_2K_3 \subseteq K$ for some submodules K_1, K_2, K_3 of M, then either $K_1K_2 \subseteq K$ or $K_3^t \subseteq K$ for some $t \geq 1$;*
4. *K is a 1-a.p submodule of M;*
5. *$(K :_R M)$ is a 1-a.p ideal of $\Re$.*

Proof. $(1) \Rightarrow (2)$ Let $K_1K_2K_3K_4 \subseteq K$ for some submodules K_1, K_2, K_3, K_4 of M. Since M is multiplication, there are ideals I_1, I_2, I_3 of $\Re$, such that $K_1 = I_1M$, $K_2 = I_2M$, $K_3 = I_3M$. Therefore, $I_1I_2I_3K_4 \subseteq K$. Proposition 5 implies that either $I_1I_2K_4 \subseteq K$ or $I_3^tK_4 \subseteq K$ for some $t \geq 1$. Thus, $K_1K_2K_4 \subseteq K$ or $K_3^tK_4 \subseteq K$ for some $t \geq 1$.

$(2) \Rightarrow (3)$ Choose $K_4 = M$.

$(3) \Rightarrow (4)$ Suppose that $I_1I_2L \subseteq K$ for some ideals I_1, I_2 of $\Re$ and some submodule L of M. It is adequate to set $K_1 = I_1M$, $K_2 = I_2M$, $K_3 = L$ in part (3).

$(4) \Rightarrow (5)$ Suppose that K is a 1-a.p submodule of M. Let nonunits $x, y, z \in \Re$ and $\eta \in M$ with $xyz \in (K :_{\Re} \eta)$. Therefore, $xy(z\eta) \in K$. Since K is a 1-a.p submodule, we obtain either $xy \in (K :_{\Re} \eta)$ or $z\eta \in M - rad(K)$. Hence, $xy \in (K :_{\Re} \eta)$ or $z \in \sqrt{(K :_{\Re} \eta)}$. Consequently, $(K :_{\Re} \eta)$ is a 1-a.p ideal of $\Re$.

$(5) \Rightarrow (1)$ Suppose that $(K :_{\Re} \eta)$ is a 1-a.p ideal of $\Re$. Let $xyz\eta \in K$ with nonunits $x, y, z \in \Re$ and $\eta \in M$. Then, $xyz \in (K :_{\Re} M)$. Since $(K :_{\Re} M)$ is a 1-a.p ideal of $\Re$, we have that $xy \in (K :_{\Re} \eta)$ or $z^t \in (K :_{\Re} \eta)$ for some $t \geq 1$. Hence, $xy\eta \in K$ or $z^t\eta \in K$. Consequently, K is a classical 1-a.p submodule of M. □

For an $\Re$-module M, as usual, the set of zero-divisors of M is denoted by $Z_{\Re}(M)$. By $Z_{\Re}(M/K)$, we denote the set of $r \in \Re$, such that $r\eta \in K$ for some $\eta \in M \backslash K$. Now, we discuss classical 1-a.p submodules of $S^{-1}M$.

Proposition 2. *Consider S as a multiplicative closed subset of a ring, $\Re$, and K as a proper submodule of M.*

1. *If K is a classical 1-a.p submodule of $\Re$-module M, such that $(K :_{\Re} M) \cap S = \emptyset$, then $S^{-1}K$ is a classical 1-a.p submodule of $S^{-1}M$.*

2. *If $S^{-1}K$ is a classical 1-a.p submodule of $S^{-1}M$ and $Z_{\Re}(M/K) \cap S = \emptyset$, then K is a classical 1-a.p submodule of M.*

Proof. (1) Let K be a classical 1-a.p submodule of M and $(K :_{\Re} M) \cap S = \emptyset$. Suppose that $\frac{a}{s}\frac{b}{t}\frac{c}{u}\frac{m}{v} \in S^{-1}K$ for some nonunits $\frac{a}{s}, \frac{b}{t}, \frac{c}{u} \in S^{-1}K$ and $\frac{m}{v} \in S^{-1}M$. Then there exists $w \in S$, such that $w(abcm) = abc(wm) \in K$. Since K is a classical 1-a.p submodule of M, we have $ab(wm) \in K$ or $c^t(wm) \in K$ for some $t \geq 1$. This implies that $\frac{abm}{stv} = \frac{abwm}{stwm} \in S^{-1}K$ or $\frac{c^t m}{uv} = \frac{c^t wm}{uwm} \in S^{-1}K$. Consequently, $S^{-1}K$ is a classical 1-a.p submodule of $S^{-1}M$.

(2) Suppose that $xyz\eta \in K$ for some nonunit elements $x, y, z \in \Re$ and $\eta \in M$. Then $\frac{x}{1}\frac{y}{1}\frac{z}{1}\frac{\eta}{1} \in S^{-1}K$, which implies $\frac{x}{1}\frac{y}{1}\frac{\eta}{1} \in S^{-1}K$ or $\left(\frac{z}{1}\right)^t\frac{\eta}{1} \in S^{-1}K$ for some $t \geq 1$. Then, $sxy\eta \in K$ for some $s \in S$ or $s'z^t\eta \in K$ for some $s' \in S$. Our assumption $Z_{\Re}(M/K) \cap S = \emptyset$ yields that $xy\eta \in K$ or $z^t\eta \in K$, as needed. □

Proposition 3. *Let $f\colon M_1 \longrightarrow M_2$ be an $\Re$-module homomorphism, and K_1 and K_2 are proper submodules of M_1 and M_2, respectively.*

1. *If K_2 is a classical 1-a.p submodule of M_2 and $f^{-1}(K_2)$ is proper, then $f^{-1}(K_2)$ is a classical 1-a.p submodule of M_1.*
2. *If f is an epimorphism and K_1 is a classical 1-a.p submodule of M_1 containing $Ker(f)$, then $f(K_1)$ is a classical 1-a.p submodule of M_2.*

Proof. (1) Suppose that K_2 is a classical 1-a.p submodule of M_2, such that $f^{-1}(K_2) \neq M_1$. Choose nonunits $x, y, z \in \Re$ and $\eta \in M_1$, such that $xyz\eta \in f^{-1}(K_2)$. Then, we have $f(xyz\eta) = xyzf(\eta) \in K_2$. As K_2 is a classical 1-a.p submodule of M_2, we conclude that $xyf(\eta) = f(xy\eta) \in K_2$ or $z^t f(\eta) = f(z^t\eta) \in K_2$ for some $t \geq 1$, which implies that $xy\eta \in f^{-1}(K_2)$ or $z^t\eta \in f^{-1}(K_2)$ for some $t \geq 1$. So, $f^{-1}(K_2)$ is a classical 1-a.p submodule of M_1.

(2) Let $xyz\acute{\eta} \in f(K_1)$ for some nonunits $x, y, z \in \Re$ and $\acute{\eta} \in M_2$. Since f is surjective, there exists $\eta \in M_1$, such that $\acute{\eta} = f(\eta)$. Then we have $xyz\acute{\eta} = f(xyz\eta) \in f(K_1)$. As K_1 contains $Ker(f)$, we have $xyz\eta \in K_1$. Since $xy\eta \in K_1$ or $z^t\eta \in K_1$ for some $t \geq 1$, then we conclude that either $xy\acute{\eta} = f(xy\eta) \in f(N_1)$ or $z^t\acute{\eta} = f(z^t\eta) \in f(K_1)$. Hence, $f(K_1)$ is a classical 1-a.p submodule of M_1. □

In view of Proposition 3, we have the subsequent corollary.

Corollary 1. *Consider M as an $\Re$-module, with $L \subseteq K$ being two of its submodules. Then K is a classical 1-a.p submodule of M if and only if K/L is a classical 1-a.p submodule of M/L.*

Recall from [21] that an $\Re$-module $\Im$ is said to be a flat $\Re$-module if, for each exact sequence $K \longrightarrow L \longrightarrow M$ of $\Re$-modules, the sequence $\Im \otimes K \longrightarrow \Im \otimes L \longrightarrow \Im \otimes M$ is also exact. Azizi in [9] (Lemma 3.2) indicated that, if M is an $\Re$-module, K is a submodule of M and $\Im$ is a flat $\Re$-module, then $(\Im \otimes K :_{\Im\otimes M} x) = \Im \otimes (K :_M x)$ for every $x \in \Re$.

Proposition 4. *Consider M as an $\Re$-module and $\Im$ as a flat $\Re$-module. If K is a classical 1-a.p submodule of M, such that $\Im \otimes K \neq \Im \otimes M$, then $\Im \otimes K$ is a classical 1-a.p submodule of $\Im \otimes M$.*

Proof. Let $x, y, z \in \Re$ be nonunits. Then, by Theorem 6, $(K :_M xyz) = (K :_M xy)$ or $(K :_M xyz) \subseteq \left(K :_M z^t\right)$ for some $t \geq 1$. Suppose that $(K :_M xyz) = (K :_M xy)$. Then, by [9] (Lemma 3.2), $(\Im \otimes K :_{\Im\otimes M} xyz) = \Im \otimes (K :_M xyz) = \Im \otimes (K :_M xy) = (\Im \otimes K :_{\Im\otimes M} xy)$. Now, assume that $(K :_M xyz) \subseteq \left(K :_M z^t\right)$ for some $t \geq 1$, then $(\Im \otimes K :_{\Im\otimes M} xyz) = \Im \otimes (K :_M xyz) \subseteq \Im \otimes \left(K :_M z^t\right) = \left(\Im \otimes K :_{\Im\otimes M} z^t\right)$. Using Theorem 6 once again, we conclude that $\Im \otimes K$ is a classical 1-a.p submodule of $\Im \otimes M$. □

Corollary 2. *Let us define M as an $\Re$-module and X as an indeterminate. If K is a classical 1-a.p submodule of M, then $K[X]$ is a classical 1-a.p submodule of $M[X]$.*

Proof. Suppose that K is a classical 1-a.p submodule of M. Observe that $\Re[X]$ is a flat $\Re$-module. So, by Theorem 4, $\Re[X] \otimes K$ is a classical 1-a.p submodule of $\Re[X] \otimes M$. Since $\Re[X] \otimes K \cong K[X]$ and $\Re[X] \otimes M \cong M[X]$, $K[X]$ is a classical 1-a.p submodule of $M[X]$ by Proposition 3. □

Let $\Re_i$ be a commutative ring with identity, M_i be an $\Re$-module, for $i = 1, 2$ and $M = M_1 \times M_2$. Then M is an $\Re$-module and it is widely recognized that each submodule K of M can be expressed as $K = K_1 \times K_2$ for some submodules K_1 of M_1 and K_2 of M_2. Now, we investigate classical 1-a.p submodules of $M = M_1 \times M_2$.

Proposition 5. *Let M_1, M_2 be $\Re$-modules and K_1, K_2 be proper submodules of M_1, M_2, respectively. If $K = K_1 \times K_2$ is a classical 1-a.p submodule of $\Re$-module $M = M_1 \times M_2$, then K_1 is a classical 1-a.p submodule of M_1 and K_2 is a classical 1-a.p submodule of M_2.*

Proof. Suppose that $K = K_1 \times K_2$ is a classical 1-a.p submodule of $M = M_1 \times M_2$. Let x, y, z be nonunits in $\Re$ and $\eta_1 \in M_1, \eta_2 \in M_2$, such that $xyz\eta \in K_1$ and $xy\eta \notin K_1$. Then $xyz(\eta, 0) \in K_1 \times K_2$ and $xy(\eta, 0) \notin K_1 \times K_2$, which implies that $z^t(\eta, 0) \in K_1 \times K_2$ for some $t \geq 1$. Thus, $z^t\eta \in K_1$ for some $t \geq 1$ and K_1 is a classical 1-a.p submodule of M_1. Similar to the argument above, K_2 is a classical 1-a.p submodule of M_2. □

We provide the subsequent example to demonstrate that the converse of Proposition 5 does not hold in general.

Example 3. *Let $\Re = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$ and $K = q\mathbb{Z} \times r\mathbb{Z}$, where q, r are two different prime integers. Since $q\mathbb{Z}, r\mathbb{Z}$ are prime ideals of $\mathbb{Z}$, then $q\mathbb{Z}, r\mathbb{Z}$ are classical 1-a.p $\mathbb{Z}-$submodules of $\mathbb{Z}$. Note that $qqr(1, 1) \in K$ but neither $qq(1, 1) \in K$ nor $r^t(1, 1) \in K$ for every $t \geq 1$. Thus, K does not qualify as a classical 1-a.p submodule of M.*

Next, we characterize the family of classical 1-a.p submodules of $\Re = \Re_1 \times \Re_2$-module $M = M_1 \times M_2$.

Proposition 6. *Let $\Re = \Re_1 \times \Re_2$ be a decomposable ring and $M = M_1 \times M_2$ be an $\Re$-module, where M_1 is an $\Re_1$-module and M_2 is an $\Re_2$-module. Then, $K = K_1 \times K_2$ is a classical 1-a.p submodule of $M = M_1 \times M_2$ if and only if one of the subsequent statements holds:*

1. *K_1 is a 1-absorbing prime submodule of M_1 and $K_2 = M_2$;*
2. *K_2 is a 1-absorbing prime submodule of M_2 and $K_1 = M_1$.*

Proof. Suppose that $K = K_1 \times K_2$ is a classical 1-a.p submodule of M. Assume that both K_1 and K_2 are proper submodules. Then there exist $\eta_1 \in M_1 \backslash K_1$ and $\eta_2 \in M_2 \backslash K_2$. Hence, $(1, 0)(1, 0)(0, 1)(\eta_1, \eta_2) \in K$, which implies either $(1, 0)(1, 0)(\eta_1, \eta_2) \in K$ or $(0, 1)^t(\eta_1, \eta_2) \in K$ for some $t \geq 1$, a contradiction. Thus, $K_1 = M_1$ or $K_2 = M_2$. We may assume without loss of generality that $K_2 = M_2$. Let $x, y, z \in \Re$ be nonunit elements and $\eta \in M_1$, such that $xyz\eta \in K_1$. Then $(x, 1)(y, 1)(z, 1)(\eta, \eta') \in K_1 \times M_2$ for all $\eta' \in M_2$. This yields that either $(x, 1)(y, 1)(\eta, \eta') \in K_1 \times M_2$ or $(z, 1)^t(\eta, \eta') \in K_1 \times M_2$ for some $t \geq 1$. Hence, $xy\eta \in K_1$ or $z^t\eta \in K_1$ for some $t \geq 1$, and so K_1 is a 1-absorbing prime submodule of M_1.

Conversely, let K_1 be a 1-absorbing prime submodule of M_1 and $K_2 = M_2$. Suppose that $(x_1, x_2)(y_1, y_2)(z_1, z_2)(\eta_1, \eta_2) \in K = K_1 \times M_2$ for some nonunits $(x_1, x_2), (y_1, y_2)$, $(z_1, z_2) \in \Re$ and $(\eta_1, \eta_2) \in M$. Then $x_1y_1z_1\eta_1 \in K_1$. Since K_1 is a classical 1-a.p submodule of M_1, we have either $x_1y_1\eta_1 \in K_1$ or $z_1^t\eta_1 \in K_1$ for some $t \geq 1$, which shows that either $(x_1, x_2)(y_1, y_2)(\eta_1, \eta_2) \in K$ or $(z_1, z_2)^t(\eta_1, \eta_2) \in K$ for some $t \geq 1$. Consequently, K is a classical 1-a.p submodule of M. □

The next example shows that if K_1 is a classical 1-a.p submodule of M_1 and K_2 is a classical 1-a.p submodule of M_2 then $K = K_1 \times K_2$ is not necessarily a classical 1-a.p submodule of $M = M_1 \times M_2$.

Example 4. *Consider $\Re = M = \mathbb{Z} \times \mathbb{Z}$ and $K = q\mathbb{Z} \times r\mathbb{Z}$, where q, r are two different prime integers. Since $q\mathbb{Z}, r\mathbb{Z}$ are prime ideals of $\mathbb{Z}$, then $q\mathbb{Z}, r\mathbb{Z}$ are 1-a.p $\mathbb{Z}-$submodules of $\mathbb{Z}$. Notice that $(q,1)(q,1)(1,r)(1,1) = (q^2, r) \in K$ but neither $(q,1)(q,1)(1,1) \in K$ nor $(1,r)^t(1,1) \in K$ for every $t \geq 1$. Consequently, K does not qualify as a classical 1-a.p submodule of M.*

We end this section showing that if there is a classical 1-a.p submodule that is not classical primary in an $\Re$-module, then $\Re$ is a local ring.

Theorem 8. *Assume M as an $\Re$-module and K as a classical 1-a.p submodule that is not classical primary. Then $\Re$ is a local ring with unique maximal ideal $\mathfrak{q}$ of $\Re$, such that $\mathfrak{q}^2 \subseteq \sqrt{(K :_\Re \eta)}$ for some $\eta \in M - K$.*

Proof. Assume that K is a classical 1-a.p submodule of M that is not a classical primary submodule. Since K is not classical primary, there exists $\eta \in M - K$, such that $(K :_\Re \eta)$ is not a primary ideal of $\Re$. On the other hand, since K is a classical 1-a.p submodule of M, then, by Theorem 1, $(K :_\Re \eta)$ is a 1-a.p ideal of $\Re$. Thus, $\Re$ admits a 1-a.p ideal that is not primary. Then by [22] (Lemma 2.1), $\Re$ is a local ring with unique maximal ideal $\mathfrak{q}$ of $\Re$, such that $\mathfrak{q}^2 \subseteq \sqrt{(K :_\Re \eta)}$ for some $\eta \in M - K$. □

3. Classical Primary-like Conditions in Amalgamated Duplication of a Module

Let $\Re$ be a ring and J an ideal of $\Re$. Recall from [18] that the amalgamated duplication (in briefly a.d) of a ring, $\Re$, along an ideal J, denoted by $\Re \bowtie J = \{(x, x + j) : x \in \Re, j \in J\}$, is a special subring of $\Re \times \Re$, with addition and multiplication performed component by component. In fact, $\Re \bowtie J$ is a commutative subring that shares the same identity element as $\Re \times \Re$. Consider M as an $\Re$-module. Recall from [20] that the a.d of an $\Re$-module M along an ideal J, denoted by $M \bowtie J = \{(\eta, \eta + \eta') : \eta \in M,\ \eta' \in JM\}$ is an $\Re \bowtie J$- module with componentwise addition and the following scalar multiplication: $(x, x + j)(\eta, \eta + \eta') = (x\eta, (x + j)(\eta + \eta'))$ for each $(x, x + j) \in R \bowtie J$ and $(\eta, \eta + \eta') \in M \bowtie J$ [20]. Observe that, if we consider $M = \Re$ as an $\Re$-module, then $M \bowtie J$ and $\Re \bowtie J$ are identical. In this section, we examine the classical primary submodules and classical 1-a.p submodules of the a.d $M \bowtie J$ of an $\Re$-module M along an ideal J. We begin with the following lemmas, which will be frequently cited throughout our main theorem (Theorem 9).

Lemma 2. *Let M be an $\Re$-module, K be a submodule of M and J be an ideal of $\Re$. Consider the $\Re \bowtie J$-module $M \bowtie J$. The subsequent conditions are satisfied.*

1. $(K \bowtie J :_{M \bowtie J} (x, x + j)) = (K :_M x) \bowtie J$ *for every $x \in \Re$ and $j \in J$;*
2. $(K \bowtie J :_{\Re \bowtie J} (\eta, \eta + \eta')) = (K :_\Re \eta) \bowtie J$ *for every $\eta \in M$ and $\eta' \in JM$.*

Proof. It can be easily verified that if K is a submodule of M then $K \bowtie J = \{(\eta, \eta + \eta') : \eta \in K, \eta' \in JM\}$ is an $\Re \bowtie J$-submodule of $M \bowtie J$.

(1) Let $(\eta, \eta + \eta') \in (K :_M \Re) \bowtie J$ for some $\eta \in M, \eta' \in JM$. Then we have $\eta \in (K :_M x)$, which implies that $x\eta \in K$. Then we infer that $(x, x + j)(\eta, \eta + \eta') = (x\eta, x\eta + x\eta' + j\eta + j\eta') \in K \bowtie J$ as $x\eta' + j\eta + j\eta' \in JM$. Hence we obtain $(\eta, \eta + \eta') \in (K \bowtie J :_{M \bowtie J} (x, x + j))$. To consider the converse, take $(\eta, \eta + \eta') \in (K \bowtie J :_{M \bowtie J} (x, x + j))$. Thus we obtain $(x, x + j)(\eta, \eta + n') \in K \bowtie J$, which leads to $x\eta \in K$, that is, $\eta \in (K :_M x)$. This gives $(\eta, \eta + \eta') \in (K :_M x) \bowtie J$, and this concludes the proof.

(2) It is comparable to (1). □

Recall the subsequent lemma from [8].

Lemma 3 ([8] (Lemma 3.1))**.** *Consider M as an $\Re$-module and K a submodule of M. Then, K is a classical prime submodule if and only if $(K :_M xy) = (K : x)$ or $(K :_M xy) = (K :_M y)$ for every $x, y \in \Re$.*

Lemma 4. *Let $\Re$ be a ring, J be an ideal of $\Re$ and consider the a.d $\Re \bowtie J$. For every $j \in J$ and $x \in \Re$, $(x, x+j)$ is a unit of $\Re \bowtie J$ if and only if x and $x+j$ are units in $\Re$.*

Proof. The "if part" is evident. For the "only if part", assume that x and $x+j$ are units in $\Re$. Then there exist $b, c \in \Re$, such that $xb = 1 = (x+j)c$. Then we conclude that $(x+j)(b-cbj) = xb - cxbj + jb - cbj^2 = xb + jb - jbc(x+j) = 1$. Then we have $(x, x+j)(b, b-cbj) = (1,1)$, which completes the proof. □

Finally, we investigate the relationship between a classical primary submodule of an $\Re$-module M and those of $\Re \bowtie J$-module $M \bowtie J$.

Theorem 9. *Let M be an $\Re$-module, K a submodule of M and J an ideal of $\Re$. Consider the $\Re \bowtie J$-module $M \bowtie J$. The subsequent conditions are satisfied.*

1. *$K \bowtie J$ is a classical primary submodule of $M \bowtie J$ if and only if K is a classical primary submodule of M;*
2. *$K \bowtie J$ is a classical 1-a.p submodule of $M \bowtie J$ if and only if K is a classical 1-a.p submodule of M.*

Proof. (1) Suppose that $K \bowtie J$ is a classical primary submodule of $M \bowtie J$. Let $xy\eta \in K$ for some $x, y \in \Re$ and $\eta \in M$. Then we have $(x,x)(y,y)(\eta,\eta) = (xy\eta, xy\eta) \in K \bowtie J$. As $K \bowtie J$ is a classical primary submodule of $M \bowtie J$, we have $(x,x)(\eta,\eta) \in K \bowtie J$ or $(y,y)^t(\eta,\eta) \in K \bowtie J$ for some $t \geq 1$, which implies that $x\eta \in K$ or $y^t\eta \in K$ for some $t \geq 1$. Hence, K is a classical primary submodule of M. For the converse, assume that K is a classical primary submodule of M. Let $(x, x+j)(y, y+i)(\eta, \eta+\eta') \in K \bowtie J$ for some $(x, x+j), (y, y+i) \in \Re \bowtie J$. Then we have $(xy\eta, (x+j)(y+i)(\eta+\acute{\eta})) \in K \bowtie J$, and so $xy\eta \in K$. Since K is a classical primary submodule, we have either $x\eta \in K$ or $y^t\eta \in K$ for some $t \geq 1$. We obtain the subsequent cases.

Case 1: Assume that $x\eta \in K$. Then $(x, x+j)(\eta, \eta+\eta') = (x\eta, x\eta + j\eta + x\eta' + j\eta') \in K \bowtie J$.

Case 2: Assume that $y^t\eta \in K$ for some $t \geq 1$.

$(y, y+i)^t(\eta, \eta+\eta') = (y^t\eta, (y+i)^t.(\eta+\acute{\eta})) = (y^t\eta, (y^t + x_1y^{t-1}i + x_2y^{t-2}i^2 + \ldots + x_ty^t)(\eta+\acute{\eta})) = (y^t\eta, (y^t + x_1y^{t-1}i + \ldots + x_ti^t)\acute{\eta} + (y^t + x_1y^{t-1}i + \ldots + x_ti^t)\eta \in K \bowtie J$.

Consequently, $K \bowtie J$ is a classical primary submodule of $M \bowtie J$.

(2) Suppose that $K \bowtie J$ is a classical 1-a.p submodule of $M \bowtie J$. Let $xyz\eta \in K$ for some nonunits $x, y, z \in \Re$ and $\eta \in M$. Then, by Lemma 4, $(x,x), (y,y), (z,z) \in \Re \bowtie J$ are nonunits and $(x,x)(y,y)(z,z)(\eta,\eta) \in K \bowtie J$. Since $K \bowtie J$ is a classical 1-a.p submodule, we have $(x,x)(y,y)(\eta,\eta) \in K \bowtie J$ or $(z,z)^t(\eta,\eta) \in K \bowtie J$ for some $t \geq 1$. This implies that $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geq 1$, that is, K is a classical 1-a.p submodule of M. For the converse, assume that K is a classical 1-a.p submodule of M. If K is a classical primary submodule of M, then, by (1), $K \bowtie J$ is a classical primary submodule of $M \bowtie J$, and so $K \bowtie J$ is a classical 1-a.p submodule of $M \bowtie J$.

Now, assume on the contrary that K is not a classical primary submodule of M. Then, by Theorem 8, $(\Re, \mathfrak{q})$ is a local ring, such that $\mathfrak{q}^2 \subseteq \sqrt{(K :_\Re \eta)}$ for some $\eta \in M - K$. Since $J \subseteq \mathfrak{q} = Jac(\Re)$, where $Jac(\Re)$ is the Jacobson radical of $\Re$, by Lemma 4, $(x, x+j)$ is a unit in $\Re \bowtie J$ if and only if x is a unit in $\Re$. Now, let $(x, x+j), (y, y+i), (z, z+k)$ be nonunits in $\Re \bowtie J$. Then we have x, y, z are nonunits in $\Re$ and $(xyz\eta, (x+j)(y+i)(z+k)(\eta+\acute{\eta})) \in K \bowtie J$. Thus, $xyz\eta \in K$. Since K is a classical 1-a.p submodule of M, we have either $xy\eta \in K$ or $z^t\eta \in K$ for some $t \geq 1$.Hence, we have subsequent cases.

Case 1: Suppose that $xy\eta \in K$. Then,

$$\begin{aligned}(x, x+j)(y, y+i)(\eta, \eta+\eta') &= (xy\eta, (x+j)(y+i)(\eta+\acute{\eta})) \\ &= (xy\eta, xy\eta + (xi + yj + ji)\eta + (xi + yj + ji)\eta') \in K \bowtie J.\end{aligned}$$

Case 2: Suppose that $z^t\eta \in K$ for some $t \geq 1$.

$$
\begin{aligned}
(z, z+k)^t(\eta, \eta+\eta') &= (z^t\eta, (z+k)^t.(\eta+\eta')) \\
&= (z^t\eta, z^t + x_1z^{t-1}i + x_2z^{t-2}i^2 + \ldots + x_ti^t)(\eta+\eta')) \\
&= (z^t\eta, (z^t + x_1z^{t-1}i + \ldots + x_ti^t)\eta' + (z^t + x_1z^{t-1}i + \ldots + x_ti^t)\eta) \in K \bowtie J.
\end{aligned}
$$

Consequently, $K \bowtie J$ is a classical 1-a.p submodule of $M \bowtie J$. □

4. Discussion and Conclusions

As prime submodules and their generalizations represent a crucial area of study in commutative algebra, numerous researchers have derived significant results utilizing various generalization methodologies. As mentioned in the introduction, some of these methods have enabled us to achieve meaningful results in our study.

Our research establishes a more extensive structure that encompasses both 1-absorbing primary submodules and classical primary submodules. In this context, we conduct a detailed examination of the properties of classical 1-absorbing primary submodules and explore their relationships with other classes of prime submodules. Furthermore, we analyze the behavior of classical 1-absorbing primary submodules within tensor product and amalgamated duplication $M \bowtie J$ of an R-module M along an ideal J. Our research indicates that the various results derived in articles [7,11,13] are applicable to classical 1-absorbing primary submodules. The properties of these submodules in different module structures have been demonstrated in their general forms, and their relationships with other submodules have been examined. Additionally, in this study, we thoroughly illustrate the relationship between 1-absorbing primary ideals in article [12] and classical 1-absorbing primary submodules within different module structures.

In conclusion, the comprehensive analysis of classical 1-absorbing primary submodules has elucidated their algebraic properties and allowed us to uncover the similarities between 1-absorbing primary ideals and submodules. To encourage future research, it is possible to define weakly classical 1-absorbing primary submodules as a generalization of classical 1-absorbing primary submodules. Investigating the relationship between this new module structure and weakly 1-absorbing primary ideals is expected to yield significant new results. Furthermore, classical 1-absorbing primary submodules can be generalized through various approaches. For instance, classical S-1 absorbing primary submodules can be studied by using S, which is a multiplicatively closed subset of the ring R with specific properties. Additionally, the ϕ function can be used to investigate ϕ-classical 1-absorbing primary submodules as an alternative generalization form of classical 1-absorbing primary submodules.

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