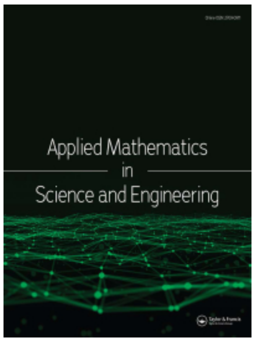

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Exploring probabilistic Bernstein polynomials: identities and applications

Ayse Karagenc[a], Mehmet Acikgoz[a] and Serkan Araci[b]

[a]Department of Mathematics, Faculty of Science and Arts, Gaziantep University, Gaziantep, Türkiye;
[b]Department of Computer Engineering, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Türkiye

ABSTRACT
In this paper, we introduce the probabilistic Bernstein polynomials and derive new and interesting correlations among several special functions and special number sequences such as Euler polynomials, Bernoulli polynomials of higher order, Frobenius–Euler polynomials of higher order, Stirling numbers of the second kind and Bell polynomials subject to several special random variables.

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1. Introduction, preliminaries and motivation

Let Y be denoted a random variable satisfying appropriate moment conditions. Adell and Lekuona [1] introduced probabilistic Stirling numbers of the second kind associated with Y. They also illustrated and characterized these numbers sequence in connection with appropriate random variables Y in each case. Ta [2] constructed Appell polynomials subject to random variables, and this approach yielded as the mean value relation, which is essential in the sense that many other relations can be acquired from it. Vellaisamy and Viens [3] considered a probabilistic approach in order to compute the Adomian polynomials and provided a probabilistic interpretation for these polynomials. Lei and Poulin [4] got several probabilistic properties of polynomials described over p-adic analysis and concluded the notion of two polynomials being strongly coprime and calculated the probability of two monic polynomials being strongly coprime. Adell [5] considered a different generalization of the Stirling numbers of the second kind associated with each complex-valued random variable satisfying appropriate integrability conditions. Also, Adell focussed on applications such as determining asymptotic behaviour without utilizing central limit theorem,

CONTACT Serkan Araci ✉ serkan.araci@hku.edu.tr Department of Computer Engineering, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep 27410, Türkiye

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and Lévy processes and cumulants. Gomaa and Magar [6] investigated a new generating function of generalized Fubini-type polynomials. By making use of these generating functions, they derived various interesting identities and relations and obtained probabilistic applications and several probabilistic properties. Frontczak and Tomovski [7] derived probabilistic interpretations for several summation formulae for generalized Bernoulli and Euler–Genocchi polynomials of order (r, m). Kim and Kim [8] also presented probabilistic variants of degenerate Stirling numbers of the second kind and the degenerate Bell polynomials. These include the probabilistic degenerate Stirling numbers of the second kind linked with Y and the probabilistic degenerate Bell polynomials related to Y. Soni *et al.* [9] built on Adell and Lekuona's work [1] to come up with a probabilistic generalization of the Bell polynomials associated with a random variable satisfying suitable moment conditions. They also gave the systematic study and applications of aforementioned polynomials in their work. Quite recently, Kim and Kim [10] introduced and studied the probabilistic extension of Bernoulli polynomials and Euler polynomials associated with Y. Also, they gave the probabilistic versions of several special polynomials such as the probabilistic two variable Fubini polynomials associated Y, r-Stirling numbers of the second kind associated Y, etc. Chen *et al.* [11] investigated the probabilistic variants of type 2 Bernoulli polynomials associated with Y and the probabilistic type 2 Euler polynomials associated with Y along with their trigonometric extensions. Xu *et al.* [12] considered a probabilistic expression of central Bell polynomials attached to Y, discovered the probabilistic central factorial numbers of the second kind attached Y and the probabilistic central Fubini polynomials attached Y. Kim *et al.* [13] studied probabilistic extensions of the degenerate Stirling polynomials of the second kind. Motivated and inspired by the works mentioned above, we explore the probabilistic extensions of Bernstein polynomials associated with Y and establish new and interesting explicit expressions, certain identities and recurrence relations among aforementioned polynomials, Euler polynomials, higher-order Bernoulli polynomials, higher-order Frobenius polynomials, Stirling numbers of the second kind and Bell polynomials.

The outline of this paper is as follows.

In Section 1, we review classical Bernstein polynomials. We remind the reader of the Euler polynomials, higher-order Bernoulli polynomials, higher-order Frobenius polynomials, Stirling numbers of the second kind and Bell polynomials. Assume that Y is a random variable with a moment-generating function, $E[e^{vY}] = \sum_{n=o}^{\infty} E[Y^n]\frac{v^n}{n!}, (|v| < r, r \in \mathbb{R}^+)$. $\{Y_j\}_{j=1}^{k}$ is a sequence of mutually independent copies of the Y with $S_k = Y_1 + Y_2 + \ldots + Y_k, (k \in \mathbb{N})$ with $S_0 = 0$. We then introduce probabilistic Bernstein polynomials associated Y, $B_{k,n}^{Y}(x)$, and higher-order probabilistic Bernoulli polynomials $\beta_{n,Y}^{(k)}(x)$, probabilistic Euler polynomials $\mathcal{E}_n^Y(x)$, probabilistic Stirling numbers of the second kind $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_Y$, Frobenius–Euler polynomials of order k $H_n^{(k)}(x|u)$ and Bell polynomials $\phi_n(x)$. We also review various probability distributions including the Poisson distribution denoted by $Y \sim Poisson(\gamma)$, Bernoulli distribution denoted by $Y \sim Bernoulli(p)$, Binomial distribution denoted by $Y \sim b(n,p)$, Geometric distribution denoted by$Y \sim Geo(p)$, Negative Binomial distribution denoted by $Y \sim NB(n,p)$ and Uniform distribution denoted by

$Y \sim U(0,1)$, which will be used in our main Equation (1) to connect several identities, and relations with some known special functions.

Our first aim is to introduce the probabilistic Bernstein polynomials by making use of the following generating function:

$$\sum_{m=l}^{\infty} B_{l,m}^{Y}(x)\frac{v^m}{m!} = \frac{(vx)^l}{l!}(\mathrm{E}[e^{Yv}])^{1-x}, (0 \leq x \leq 1, l = 0, 1, 2, \ldots, m), \tag{1}$$

which we are going to study on this new family of polynomials.

In Section 2, we establish Theorem 2.1 regarding the relationship between probabilistic Bernstein polynomials and probabilistic Stirling numbers of the second kind, as well as their relationship between probabilistic Euler polynomials as Theorem 2.2 and probabilistic Bernoulli polynomials of higher order as Theorem 2.4. We show that Theorem 2.3 includes the probabilistic Stirling numbers of the second kind and probabilistic Bernstein polynomials that can be associated with Y have a perspective power function x^k. As for Theorem 2.5 and Theorem 2.6, the results were obtained with respect to t and x respectively.

In Section 3, we conclude the paper with the probabilistic Bernstein polynomials associated with appropriate several special random variables Y such as Poisson, Bernoulli, Binom, Geometric, Negative Binom and Uniform distributions.

Bernstein basis polynomials (or called Bernstein polynomials) used in terms of Bernstein operators have many applications and results in the approximation theory, numerical analysis, combinatorics, p-adic analysis, q-analysis, etc. *cf.* [14,15].

Throughout of this paper, we use the following standarts notations:

$$\mathbb{N} := \{1, 2, 3, \ldots\}, \ \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}.$$

As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ will, respectively, be denoted integers, rational numbers, reel numbers and complex numbers.

The generating function of Bernstein polynomials is introduced by Acikgoz and Araci [14] as follows:

Let $x \in [0,1], m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $t \in \mathbb{C}$

$$F_l(x, v) = \frac{(vx)^l}{l!}e^{v(1-x)} = \sum_{m=l}^{\infty} B_{l,m}(x)\frac{v^m}{m!}, \tag{2}$$

where $l = \overline{0, m} := 0, 1, 2, \cdots, m$ and

$$B_{l,m}(x) = \left(\begin{array}{c} m \\ l \end{array} \right) x^l(1-x)^{m-l}. \tag{3}$$

Bernstein polynomials of degree m can be written as the sum of two Bernstein polynomials of degree $(m-1)$:

$$B_{l,m}(x) = (1-x)B_{l,m-1}(x) + xB_{l-1,m-1}(x). \tag{4}$$

The derivative of Bernstein polynomials of degree m is written as a linear combination of Bernstein polynomials of degree $(m-1)$ as

$$\frac{d}{dx}B_{l,m}(x) = m(B_{l-1,m-1}(x) - B_{l,m-1}(x)). \tag{5}$$

For further information on Bernstein polynomials, see [14,15].

Let $B_n(x)$ be the Bernoulli polynomials given by

$$\tfrac{v}{e^v-1}e^{xv}=\sum_{n=0}^{\infty}B_n(x)\tfrac{v^n}{n!},(|v|<2\pi), \tag{6}$$

where if we take $x=0$ in (6), we have $B_n(0):=B_n$ that stands for Bernoulli numbers, see [7,10,11,16,17].

In [16], the Bernoulli polynomials of higher order (or order α) are known as:

$$\left(\tfrac{v}{e^v-1}\right)^{\alpha}e^{xv}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\tfrac{v^n}{n!},(|v|<2\pi). \tag{7}$$

In [7,10,11,17–19], the Euler polynomials are defined by

$$\tfrac{2}{e^v+1}e^{xv}=\sum_{n=0}^{\infty}E_n(x)\tfrac{v^n}{n!},(|v|<\pi), \tag{8}$$

where if we take $x=0$ in (8), we have $E_n(0):=E_n$ that means Euler numbers.

In [1,5,13,19], the Stirling numbers of the second kind are defined by

$$\sum_{m=l}^{\infty}\left\{\begin{matrix} m \\ l \end{matrix}\right\}\tfrac{v^m}{m!}=\tfrac{(e^v-1)^l}{l!}. \tag{9}$$

In [8], Bell polynomials $\phi_n(x)$ are defined by the help of generating function:

$$\sum_{n=0}^{\infty}\phi_n(x)\tfrac{v^n}{n!}=e^{x(e^v-1)}, \tag{10}$$

which overlaps with the moment generating function for Poisson distribution having the mean x.

In [20], Frobenius–Euler polynomials of order $k(\in\mathbb{R})$ are known as

$$\left(\tfrac{1-u}{e^v-u}\right)^{k}e^{xv}=\sum_{n=0}^{\infty}H_n^{(k)}(x|u)\tfrac{v^n}{n!},(u\in\mathbb{C}-\{1\}). \tag{11}$$

Let Y be chosen as a random variable satisfying the moment conditions by

$$\mathrm{E}[|Y|^n]<\infty(n\in\mathbb{N}_0)\text{ and }\lim_{n\to\infty}\frac{|v|^n\mathrm{E}[|Y|^n]}{n!}=0,(|v|<r;r>0),$$

where E means mathematical expectation. From here, one may write (*cf.* [1]) that

$$\mathrm{E}[e^{ivY}]=\sum_{n=o}^{\infty}\mathrm{E}[Y^n]i^n\tfrac{v^n}{n!},(|v|<r). \tag{12}$$

Equivalently, (see (*cf.* [1]))

$$\mathrm{E}[e^{v|Y|}]<\infty,|v|<r.$$

Let $\{Y_j\}_{j=1}^{k}=(Y_1,Y_2,\cdots,Y_k)$ be a sequence of mutually independent pickes of Y and be denoted by

$$S_k=Y_1+Y_2+\ldots+Y_k,(k\in\mathbb{N}) \tag{13}$$

with the initial assumption $S_0=0$.

Several distributions with probability density function, which will be used in the final section of this paper, are listed below (see [4,8,10,13,21,22]):

Poisson distribution. $Y \sim Poisson(\gamma)$ with the parameter $\gamma > 0$ with the probability density function (pdf) $P(Y=y) = f(y) = \frac{e^{-\gamma}\gamma^{y}}{y!}$, $(y \in \mathbb{N}_0)$ yielding moment generating function (mgf) as

$$\mathrm{E}[e^{vY}] = e^{\gamma\,(e^{v}-1)}. \tag{14}$$

Bernoulli distribution. $Y \sim Bernoulli(p)$ with the success probability p with pdf $f(y) = p^{y}(1-p)^{1-y}$, $(y = 0, 1)$ yielding mgf

$$\mathrm{E}[e^{Yv}] = q + pe^{v}, p + q = 1. \tag{15}$$

Binomial distribution. $Y \sim b(n,p)$ with pdf $f(y) = \left(\begin{array}{c} n \\ y \end{array}\right) p^{y}q^{n-y}$, $(y = 0, 1, 2, \cdots, n)$ yielding mgf

$$\mathrm{E}[e^{Yv}] = (pe^{v} + q)^{n}, p + q = 1. \tag{16}$$

Geometric distribution. $Y \sim Geo(p)$ with pdf $f(y) = pq^{y-1}$, $(y = 1, 2, \cdots)$ yielding mgf

$$\mathrm{E}[e^{Yv}] = \frac{pe^{v}}{1 - qe^{v}}, p + q = 1. \tag{17}$$

Negative Binomial distribution. $Y \sim NB(n,p)$ with pdf

$$f(y, r, p) = \left(\begin{array}{c} y-1 \\ r-1 \end{array}\right) p^{r}q^{y-r}, y = r, r+1, \ldots \tag{18}$$

yielding mgf

$$\mathrm{E}[e^{Yv}] = \left(\tfrac{pe^{v}}{1-qe^{v}}\right)^{r}, \left(v < \ln\tfrac{1}{1-p}\right). \tag{19}$$

Uniform distribution. Let $Y \sim U(0,1)$ be continue random variable with pdf

$$\begin{array}{c} f(y) = 1, 0 \leq y \leq 1 \\ \mathrm{E}[e^{Yv}] = \frac{e^{v}-1}{v}. \end{array} \tag{20}$$

In [1], Adell and Lekuona gave the probabilistic Stirling numbers of the second kind associated with Y are defined by

$$\frac{(\mathrm{E}[e^{Yv}]-1)^{k}}{k!} = \sum_{n=k}^{\infty} \left\{\begin{array}{c} n \\ k \end{array}\right\}_{Y} \frac{v^{n}}{n!}. \tag{21}$$

Their approach opened a new era to construct the generating function of some special functions and polynomials in the case when e^{v} changed by $E[e^{vY}]$. This approach leverages the moment generating function to analyse special functions and polynomials, enabling a deeper investigation of their behaviour within probability theory.

For example, Kim and Kim [10] introduced a new family of Bernoulli polynomials subject to a random variable Yas follows:

$$\frac{v}{\mathrm{E}[e^{Yv}]-1}(\mathrm{E}[e^{Yv}])^x = \sum_{n=0}^{\infty} \beta_n^Y(x)\frac{v^n}{n!}, (n \geq 0), \tag{22}$$

where $\beta_n^Y(x)$ are called probabilistic Bernoulli polynomials. In the case when $Y = 1, \beta_n^1(x) := B_n(x)$ turns out to be well-known (classical or ordinary) Bernoulli polynomials. Also, at the value of $x = 0, \beta_n^Y(0) = \beta_n^Y$ are called the probabilistic Bernoulli numbers.

In the same manner, one may consider the probabilistic Bernoulli polynomials of higher order (or order k) associated with Y by

$$\left(\frac{v}{\mathrm{E}[e^{Yv}]-1}\right)^k(\mathrm{E}[e^{Yv}])^x = \sum_{n=0}^{\infty} \beta_{n,Y}^{(k)}(x)\frac{v^n}{n!}. \tag{23}$$

Kim and Kim [10] defined probabilistic Euler polynomials associated with Y by

$$\frac{2}{\mathrm{E}[e^{Yv}]+1}(\mathrm{E}[e^{Yv}])^x = \sum_{n=0}^{\infty} \mathcal{E}_n^Y(x)\frac{v^n}{n!}. \tag{24}$$

In the case when $Y = 1, \mathcal{E}_n^1(x) := \mathcal{E}_n(x), (n \geq 0)$. At the value of $x = 0$, $\mathcal{E}_n^Y(0) = \mathcal{E}_n^Y$ are called the probabilistic Euler numbers associated with Y.

Motivated and inspired by above, we introduce the generating function of the probabilistic Bernstein polynomials as follows:

$$\sum_{m=l}^{\infty} B_{l,m}^Y(x)\frac{v^m}{m!} = \frac{(vx)^l}{l!}(\mathrm{E}[e^{Yv}])^{1-x}, (0 \leq x \leq 1, l = 0, 1, 2, \dots, m).$$

By making use of the generating function of the probabilistic Bernstein polynomials, we derive some new interesting identities and relations. We consider a random variable Y as a Poisson, Bernoulli, Binom, Geometric, Negative Binomial and Uniform random variable, and then we get the new correlations among the probabilistic Bernstein polynomials, Bernoulli polynomials of higher order, Bell polynomials, Stirling numbers of the second kind, Frobenius–Euler polynomials of higher order and Euler polynomials.

2. Introducing probabilistic Bernstein polynomials with their certain identities

In this section, we are now in a position to state the following Theorems, which contain probabilistic Stirling numbers of the second kind, probabilistic Euler polynomials, recurrence relation, and derivative relation in terms of probabilistic Bernstein polynomials.

Theorem 2.1: *Let Y be a random variable, and $m \in \mathbb{N}_0$ with $l = \overline{0, m}$. Then we have the explicit identity:*

$$B_{l,m}^Y(x) = \sum_{j=0}^{m-l} \binom{m}{l} x^l (1-x)_j \left\{ \begin{matrix} m-l \\ j \end{matrix} \right\}_Y.$$

Proof: By (1), we have

$$\sum_{m=l}^{\infty} B^{Y}_{l,m}(x)\frac{v^m}{m!} = \frac{(vx)^l}{l!}(\mathrm{E}[e^{Yv}]-1+1)^{1-x} = \frac{(vx)^l}{l!}\sum_{j=0}^{\infty}\binom{1-x}{j}(\mathrm{E}[e^{Yv}]-1)^j$$

$$= \frac{(vx)^l}{l!}\sum_{j=0}^{\infty}\frac{(1-x)_j}{j!}(\mathrm{E}[e^{Yv}]-1)^j$$

$$= \frac{(vx)^l}{l!}\sum_{j=0}^{\infty}(1-x)_j\sum_{m=j}^{\infty}\left\{ \begin{matrix} m \\ j \end{matrix} \right\}_Y \frac{v^m}{m!}$$

$$= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m+l}{l}x^l(1-x)_j\left\{ \begin{matrix} m \\ j \end{matrix} \right\}_Y\right)\frac{v^{m+l}}{(m+l)!}$$

$$= \sum_{m=l}^{\infty}\left(\sum_{j=0}^{m-l}\binom{m}{l}x^l(1-x)_j\left\{ \begin{matrix} m-l \\ j \end{matrix} \right\}_Y\right)\frac{v^m}{m!},$$

thus, we complete the proof. ■

Theorem 2.2: *Let Y be a random variable, and $n \in \mathbb{N}_0$ with $k = \overline{0,n}$. Then we have*

$$\binom{n}{k}x^k\mathcal{E}^{Y}_{n-k}(1) = \sum_{l=0}^{n}\binom{n}{l}\mathcal{E}^{Y}_{n-l}(x)B^{Y}_{k,l}(x).$$

Proof: Since

$$\frac{(tx)^k}{k!}(\mathrm{E}[e^{Yt}])^{1-x} = \sum_{n=k}^{\infty}B^{Y}_{k,n}(x)\frac{t^n}{n!},$$

then we compute

$$\frac{(tx)^k}{k!}\mathrm{E}[e^{Yt}] = (\mathrm{E}[e^{Yt}])^x\sum_{n=k}^{\infty}B^{Y}_{k,n}(x)\frac{t^n}{n!}$$

$$\frac{2}{\mathrm{E}[e^{Yt}]+1}\frac{(tx)^k}{k!}\mathrm{E}[e^{Yt}] = \frac{2(\mathrm{E}[e^{Yt}])^x}{\mathrm{E}[e^{Yt}]+1}\sum_{n=k}^{\infty}B^{Y}_{k,n}(x)\frac{t^n}{n!}$$

$$\left(\sum_{n=0}^{\infty}\mathcal{E}^{Y}_{n}(1)\frac{t^n}{n!}\right)\frac{(tx)^k}{k!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}^{Y}_{n-l}(x)B^{Y}_{k,l}(x)\right)\frac{t^n}{n!}$$

$$\sum_{n=k}^{\infty}\binom{n}{k}x^k\mathcal{E}^{Y}_{n-k}(1)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}^{Y}_{n-l}(x)B^{Y}_{k,l}(x)\right)\frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$ on both sides of the above, we arrive at the desired result. ■

Theorem 2.3: *Let Y be a random variable, and $n \in \mathbb{N}_0$ with $k = \overline{0,n}$. Then we have power function x^k in terms of probabilistic Stirling numbers of the second kind associated with Y and probabilistic Bernstein polynomials associated with Y as follows:*

$$x^k = \frac{1}{\binom{n}{k} E[Y^{n-k}]} \sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \left\{ \begin{matrix} l \\ m \end{matrix} \right\}_Y (x)_m B^Y_{k,n-l}(x).$$

Proof: We first consider the alternative form of the generating function of the probabilistic Bernstein polynomials by

$$\frac{(tx)^k}{k!} \mathrm{E}[e^{Yt}] = (\mathrm{E}[e^{Yt}])^x \sum_{n=0}^{\infty} B^Y_{k,n}(x) \frac{t^n}{n!}.$$

By applying elemantary calculations to the above, then we readily see that

$$\frac{(tx)^k}{k!} \sum_{n=0}^{\infty} \mathrm{E}[Y^n] \frac{t^n}{n!} = (\mathrm{E}[e^{Yt}] - 1 + 1)^x \sum_{n=0}^{\infty} B^Y_{k,n}(x) \frac{t^n}{n!}$$

$$\sum_{n=k}^{\infty} x^k \binom{n}{k} \mathrm{E}[Y^{n-k}] \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{m=0}^{l} \binom{n}{l} \left\{ \begin{matrix} l \\ m \end{matrix} \right\}_Y (x)_m B^Y_{k,n-l}(x) \right) \frac{t^n}{n!}.$$

On comparing the coefficients of t^n, we conclude the proof. ■

Theorem 2.4: *Let Ybe a random variable, and $n \in \mathbb{N}_0$ with $k = \overline{0,n}$. Then the probabilistic Bernstein polynomials subject to Y can be written as the bilateral binomial summation of the product of the probabilistic Bernoulli polynomials of higher order associated with Yand the probabilistic Bernstein polynomials associated with Y:*

$$B^Y_{k,n}(x) = x^k \sum_{l=0}^{n-k} \beta^{(k)}_{n-k-l,Y}(xk) \sum_{m=0}^{l} (1-x-xk)_m \left\{ \begin{matrix} l+k \\ m+k \end{matrix} \right\}_Y \binom{m+k}{k} \binom{n}{l+k}.$$

Proof: We may first consider the generating function of the probabilistic Bernstein polynomials associated with Y in this form:

$$\frac{(tx)^k}{k!} (\mathrm{E}[e^{Yt}])^{1-x} = x^k \left(\frac{(\mathrm{E}[e^{Yt}])^x t}{\mathrm{E}[e^{Yt}] - 1} \right)^k \frac{(\mathrm{E}[e^{Yt}] - 1)^k}{k!} (\mathrm{E}[e^{Yt}])^{1-x-xk}.$$

Then it becomes

$$= x^k \left(\sum_{n=0}^{\infty} \beta^{(k)}_{n,Y}(xk) \frac{t^n}{n!} \right) \left(\frac{(E[e^{Yt}] - 1)^k}{k!} (E[e^{Yt}] - 1 + 1)^{1-x-xk} \right).$$

By making some elemantary manipulations over the series, we see that

$$= \left(x^k \sum_{n=0}^{\infty} \beta^{(k)}_{n,Y}(xk) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \frac{(1-x-xk)_m}{m!k!} (\mathrm{E}[e^{Yt}] - 1)^{m+k} \right)$$

$$= \left(x^k \sum_{n=0}^{\infty} \beta^{(k)}_{n,Y}(xk) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} (1-x-xk)_m \binom{m+k}{k} \sum_{n=m}^{\infty} \left\{ \begin{matrix} n \\ m+k \end{matrix} \right\}_Y \frac{t^n}{n!} \right)$$

$$= \sum_{n=k}^{\infty} \left(x^k \sum_{l=0}^{n-k} \beta^{(k)}_{n-k-l,Y}(xk) \sum_{m=0}^{l} (1-x-xk)_m \left\{ \begin{matrix} l+k \\ m+k \end{matrix} \right\}_Y \binom{m+k}{k} \binom{n}{l+k} \right) \frac{t^n}{n!},$$

which yields the proofs when compared the coefficients of $\frac{t^n}{n!}$. ■

Theorem 2.5: *Let Y be a random variable, and $n \in \mathbb{N}_0$ with $k = \overline{0,n}$. The probabilistic Bernstein polynomials can be written as blending together two probabilistic Bernstein polynomials by*

$$B^Y_{k,n}(x) = xB^Y_{k-1,n-1}(x) + (1-x)E[Y]B^Y_{k,n-1}(x).$$

Proof: Taking the partial derivative for the Equation (1) with respect to the variable t, then

$$\sum_{n=k}^{\infty} B^Y_{k,n}(x) \frac{t^{n-1}}{(n-1)!} = x\frac{(tx)^{k-1}}{(k-1)!}(\mathrm{E}[e^{Yt}])^{1-x} + (1-x)\mathrm{E}[Y]\frac{(tx)^k}{k!}(\mathrm{E}[e^{Yt}])^{1-x}.$$

■

When putting the definition of the probabilistic Bernstein polynomials into the right hand side of the above, and when compared the coefficients t^n, we arrive at the desired result.

Before the next Theorem, we review the following identity:

$$\log \mathrm{E}[e^{Yt}] = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(\mathrm{E}[e^{Yt}]-1)^j}{j}$$

$$= \sum_{l=1}^{\infty} (-1)^{l-1} \left(\sum_{n=l}^{\infty} \left\{ \begin{matrix} n \\ l \end{matrix} \right\}_Y \frac{t^n}{n!} \right)$$

$$= \sum_{l=0}^{\infty} (-1)^l \sum_{n=l}^{\infty} \left\{ \begin{matrix} n+1 \\ l+1 \end{matrix} \right\}_Y \frac{t^{n+1}}{(n+1)!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} (-1)^l \left\{ \begin{matrix} n+1 \\ l+1 \end{matrix} \right\}_Y \right) \frac{t^{n+1}}{(n+1)!}.$$

Theorem 2.6: *Let Y be a random variable, and $n \in \mathbb{N}_0$ with $k = \overline{0,n}$. The classical derivatives of the probabilistic Bernstein polynomials are obtained by*

$$\frac{d}{dx}B^Y_{k,n}(x) = nB^Y_{k-1,n-1}(x) - \sum_{m=0}^{n-1} \sum_{l=0}^{m} \binom{n}{m+1} B^Y_{k,n-1-m}(x)(-1)^l \left\{ \begin{matrix} m+1 \\ l+1 \end{matrix} \right\}_Y.$$

Proof: In the generating function of the probabilistic Bernstein polynomials, when we apply the derivative operator with respect to x, it yields

$$= t\frac{(tx)^{k-1}}{(k-1)!}(\mathrm{E}[e^{Yt}])^{1-x} - \frac{(tx)^k}{k!}(\mathrm{E}[e^{Yt}])^{1-x}\log \mathrm{E}[e^{Yt}]$$

$$= \sum_{n=k}^{\infty} tB_{k-1,n}^{Y}(x)\frac{t^{n-1}}{(n-1)!} + \sum_{n=k}^{\infty} B_{k,n}^{Y}(x)\frac{t^n}{n!}\log \mathrm{E}[e^{Yt}].$$

From here, it follows

$$\sum_{n=k}^{\infty} B_{k-1,n-1}^{Y}(x)\tfrac{t^n}{(n-1)!} + \sum_{n=0}^{\infty} B_{k,n}^{Y}(x)\tfrac{t^n}{n!}\log \mathrm{E}[e^{Yt}] = \sum_{n=k}^{\infty} B_{k-1,n-1}^{Y}(x)\tfrac{t^n}{(n-1)!}$$
$$+ \sum_{n=0}^{\infty} B_{k,n}^{Y}(x)\tfrac{t^n}{n!}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}(-1)^l\left\{ \begin{array}{c} n+1 \\ l+1 \end{array}\right\}_Y\right)\tfrac{t^{n+1}}{(n+1)!}$$

$$= \sum_{n=k}^{\infty} B_{k-1,n-1}^{Y}(x)\frac{t^n}{(n-1)!}$$
$$+ \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n+1}{m+1}B_{k,n-m}^{Y}(x)(-1)^l\left\{ \begin{array}{c} m+1 \\ l+1 \end{array}\right\}_Y\right)\frac{t^{n+1}}{(n+1)!}.$$

On comparing the coefficients on the above, we get the proof of this theorem. ■

3. Applications of probabilistic Bernstein polynomials in special random variables *Y*

In this section, utilizing appropriate special random variables, e.g. Poisson, Bernoulli, binomial, geometric, negative binomial and uniform distributions, we evaluate the probabilistic Bernstein polynomials, and then we derive not only explicit identities and integral representatios (bosonic, fermionic p-adic integrals and Riemann integral sense) but also new correlations among Bell polynomials, Stirling numbers of the second kind, Frobenius–Euler numbers of higher order, Bernoulli polynomials of higher order.

Firstly, we begin with choosing $Y \sim Poisson(\alpha)$ with the parameter $\alpha > 0$ by the following Theorem.

Theorem 3.1: *The probabilistic Bernstein polynomials associated with the Poisson random variable Y can be written in terms of Bell polynomials as* $B_{k,n}^{Y}(x) = \binom{n}{k} x^k \phi_{n-k}(\alpha(1-x))$.

***Proof:*:** Since

$$\mathrm{E}[e^{Yt}] = e^{\alpha(e^t-1)},$$

we have

$$\sum_{n=k}^{\infty} B_{k,n}^{Y}(x)\frac{t^n}{n!} = \frac{(tx)^k}{k!}e^{\alpha(1-x)(e^t-1)}$$

$$= \frac{(tx)^k}{k!}\sum_{n=0}^{\infty}\phi_n(\alpha(1-x))\frac{t^n}{n!}$$

$$= \sum_{n=k}^{\infty}\left(\begin{array}{c} n \\ k \end{array}\right)x^k\phi_{n-k}(\alpha(1-x))\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, we arrive at the desired result. ■

Also, in the case of that Y is poisson random variable, $Y \sim Poisson(\alpha)$, we have the following corollaries which can be obtained by series manipulations.

Corollary 3.1: *The following relation holds true:*

$$B_{l,m}^{Y}(x) = \sum_{\tau=0}^{m-l}\binom{m}{l}\alpha^{\tau}x^{l}(1-x)^{\tau}\left\{\begin{matrix} m-l \\ \tau \end{matrix}\right\}.$$

Corollary 3.2: *Taking Riemann integral of both sides of the Corollary (3.1), we have*

$$\int_0^1 B_{l,m}^{Y}(x)dx = \sum_{\tau=l}^{m-l}\frac{\binom{m}{l}\alpha^{\tau-l}}{\binom{\tau}{l}(\tau+1)}\left\{\begin{matrix} m-l \\ \tau-l \end{matrix}\right\}.$$

Corollary 3.3: *The following relation holds true:*

$$B_{l,m}^{Y}(x) = \sum_{\tau=0}^{m-l}\binom{m}{l}\alpha^{\tau}x^{l}(1-x)^{\tau}B_{m-l,\tau}(1,1,\ldots,1).$$

Corollary 3.4: *Taking Riemann integral of both sides of the Corollary (3.3), we have*

$$\int_0^1 B_{l,m}^{Y}(x)dx = \sum_{\tau=l}^{m-l}\frac{\left(\begin{array}{c} m \\ l \end{array}\right)\alpha^{\tau-l}}{\left(\begin{array}{c} \tau \\ l \end{array}\right)(\tau+1)}B_{m-l,\tau-l}(1,1,\ldots,1).$$

Let $Y \sim Bernoulli(p)$ with the success probability p. Then, we have the following theorem.

Theorem 3.2: *The probabilistic Bernstein polynomials associated with the Bernoulli random variable Y can be written in terms of linear combination of Stirling numbers of the second kind as* $B_{l,m}^{Y}(x) = x^{l} \left(\begin{array}{c} m \\ l \end{array} \right) \sum_{\tau=0}^{m-l} p^{\tau} (1-x)_{\tau} \left\{ \begin{array}{c} m-l \\ \tau \end{array} \right\}$.

Proof: From (15), we see that

$$\sum_{m=l}^{\infty} B_{l,m}^{Y}(x) \frac{t^{m}}{m!} = \frac{(tx)^{l}}{l!} (p(e^{t}-1)+1)^{1-x}$$

$$= \frac{(tx)^{k}}{k!} \sum_{\tau=0}^{\infty} p^{\tau} \frac{(1-x)_{\tau}}{\tau!} (e^{t}-1)^{\tau}$$

$$= \sum_{m=l}^{\infty} \left(x^{l} \left(\begin{array}{c} m \\ l \end{array} \right) \sum_{\tau=0}^{m-l} p^{\tau} (1-x)_{\tau} \left\{ \begin{array}{c} m-l \\ \tau \end{array} \right\} \right) \frac{t^{m}}{m!},$$

in the case of comparing coefficients $\frac{t^{m}}{m!}$ on the both sides of the above, we conclude the proof of this theorem. ■

Let $Y \sim b(n,p)$ be a binomial random variable with success p. Then, we have the following Theorem.

Theorem 3.3: *The probabilistic Bernstein polynomials associated with the Binomials random variable Y can be written in terms of a linear combination of Stirling numbers of the second kind as*

$$B_{l,m}^{Y}(x) = x^{l} \left(\begin{array}{c} m \\ l \end{array} \right) \sum_{j=0}^{m-l} p^{j} ((1-x)(m-l))_{j} \left\{ \begin{array}{c} m-l \\ j \end{array} \right\}.$$

Proof: It can be written from (16) that

$$\sum_{m=l}^{\infty} B_{l,m}^{Y}(x) \frac{v^{m}}{m!} = \frac{(vx)^{l}}{l!} (p(e^{v}-1)+1)^{(1-x)m}$$

$$= \frac{(vx)^{l}}{l!} \sum_{j=0}^{\infty} \left(\begin{array}{c} m-mx \\ j \end{array} \right) (p(e^{v}-1))^{j}$$

$$= \frac{(vx)^{l}}{l!} \sum_{j=0}^{\infty} p^{j} (m-mx)_{j} \sum_{m=j}^{\infty} \left\{ \begin{array}{c} m \\ j \end{array} \right\} \frac{v^{m}}{m!}$$

$$= \sum_{m=l}^{\infty} \left(x^{l} \left(\begin{array}{c} m \\ l \end{array} \right) \sum_{j=0}^{m-l} p^{j} ((1-x)(m-l))_{j} \left\{ \begin{array}{c} m-l \\ j \end{array} \right\} \right) \frac{v^{m}}{m!},$$

which asserts the proof of this Theorem. ■

Let $Y \sim Geo(p)$ be a geometric distribution with obtaining the first success p.

Theorem 3.4: *The probabilistic Bernstein polynomials associated with the Geometric random variable Y can be written in terms of Frobenius–Euler number of higher order as*

$$(-1)^{m-l}B_{l,m}^{Y}(x) = x^{l}\left(\begin{array}{c} m \\ l \end{array}\right)H_{m-l}^{(1-x)}(q).$$

Proof: Since

$$\sum_{m=l}^{\infty} B_{l,m}^{Y}(x)\frac{v^m}{m!} = \frac{(vx)^l}{l!}\left(\frac{pe^v}{1-qe^v}\right)^{1-x}$$

$$= \frac{(vx)^l}{l!}\sum_{m=0}^{\infty} H_m^{(1-x)}(q)\frac{(-v)^m}{m!}$$

$$= \sum_{m=l}^{\infty} x^l\left(\begin{array}{c} m \\ l \end{array}\right)H_{m-l}^{(1-x)}(q)(-1)^{m-l}\frac{v^m}{m!},$$

we complete the proof when compared the coefficients $\frac{t^n}{n!}$ on the above. ■

We now review some basic facts on bosonic p-adic integral and fermionic integral over $\mathbb{Z}_p$. Firstly, p will be considered as a fixed prime numer for bosonic p-adic integral and as an odd prime number for fermionic p-adic integrals over $\mathbb{Z}_p$ being the ring of p-adic rational integers. see [18,23,24]. Let $C(\mathbb{Z}_p)$ be the space of all continuous functions on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim [18]:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{n\to\infty}\sum_{k=0}^{p^n-1}(-1)^k f(k). \tag{25}$$

From here, one has

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0)$$

where $f_1(x) = f(x+1)$. From (25), one can easily see that

$$E_k = \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x),$$

which is well-known as a Witt's formula, cf. [18].

Let $C(\mathbb{Z}_p)$ be the space of continuous on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$is defined by (see [23]):

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{k=0}^{p^N-1} f(k). \tag{26}$$

Let $f_1(x)$ be a translational function as $f(x+1)$. Then, we have the following difference equation as follows:

$$I(f_1) - I(f) = f'(0), \textit{see } [8].$$

Set $f(x) = e^{xt}$ in (26), one can easily see that

$$B_k = \int_{\mathbb{Z}_p} x^k d\mu(x).$$

An immediate result of Theorem (3.4) when taken bosonic and fermionic p-adic integral both sides, respectively, of the above equations gives.

Corollary 3.5: *The following identity holds true:*

$$\int_{\mathbb{Z}_p} \frac{B^Y_{k,n}(x)}{H^{(1-x)}_{n-k}(q)} d\mu(x) = (-1)^{n-k} \binom{n}{k} B_k$$

and

$$\int_{\mathbb{Z}_p} \frac{B^Y_{k,n}(x)}{H^{(1-x)}_{n-k}(q)} d\mu(x) = (-1)^{n-k} \binom{n}{k} E_k.$$

By the same manner as Geometric random variable, we obtain the following theorem in the case of using Negative binomial random variable, $Y \sim NB(n, p)$.

Theorem 3.5: *The probabilistic Bernstein polynomials associated with the Negative binomial random variable Ycan be written in terms of summation of the products of Frobenius–Euler number of higher order and Bernstein basis polynomials as*

$$B^Y_{l,m}(x) = \sum_{j=0}^{m} \binom{m}{j} B_{l,j}(x) H^{(r(1-x))}_{m-j}(q^{-1}) r^{m-l}.$$

Finally, let $Y \sim U(0,1)$ be a continue uniform random variable. Then, we conclude the paper with the following Theorem.

Theorem 3.6: *The probabilistic Bernstein polynomials associated with the continue uniform random variable Ycan be written in terms of Bernoulli polynomials of higher order as*

$$B^Y_{l,m}(x) = x^l \binom{m}{l} B^{(x-1)}_{m-l}.$$

Proof: By making use of

$$\sum_{m=l}^{\infty} B_{l,m}^{Y}(x)\frac{v^m}{m!} = \frac{(vx)^l}{l!}\left(\frac{v}{e^v-1}\right)^{x-1}$$

$$= \sum_{m=l}^{\infty} x^l \binom{m}{l} B_{m-l}^{(x-1)}\frac{v^m}{m!}.$$

Thus, we conclude this theorem with the series manipulation as above. ■

4. Conclusion

In the paper, we have introduced a new generating function of probabilistic Bernstein polynomials and given the systematic works for them. By making use of this generating function, we have derived some new relations. Also, in the case of that given random variable was Bernoulli, Binom, Negative Binom, Poisson, Geometric and Uniform distributions, we have seen that we could construct many links between well-known special polynomials such as Bernoulli polynomials of higher order, Euler polynomials, Frobenius–Euler polynomials of higher order and probabilistic Stirling numbers of the second kind.

Seemingly that these types of works would be studied for other types of special functions and polynomials.

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