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ORIGINAL RESEARCH

On the Dirichlet's type of Eulerian polynomials

Serkan Araci · Mehmet Acikgoz · Erkan Ağyüz

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Abstract In the present paper, we introduce the Eulerian polynomials attached to χ using p-adic q-integral on $\mathbb{Z}_p$. Also, we give some new interesting identities via the generating functions of Dirichlet's type of Eulerian polynomials. In addition, by applying Mellin transformation to the generating function of Dirichlet's type of Eulerian polynomials, we define Eulerian L type function which interpolates Dirichlet's type of Eulerian polynomials at negative integers.

Keywords Eulerian polynomials · p-adic q-integral on $\mathbb{Z}_p$ · Mellin transformation · L function

Mathematics Subject Classification 11S80 · 11B68

Introduction

Recently, Kim et al. have studied on the Eulerian polynomials and derived Witt's formula for the Eulerian polynomials together with the relations between Genocchi, Tangent and Euler numbers. For more on this and related issues, see, e.g., [1]. Looking at the arithmetic works of T. Kim, Y. Simsek, H. M. Srivastava and other related mathematicians, they have introduced many various generating functions for types of the Bernoulli, the Euler, the Genocchi numbers and polynomials and derived some new interesting identities (see [1–28] for a systematic work).

Kim originally defined the p-adic integral on $\mathbb{Z}_p$ based on the q-integers (called p-adic q-integral on $\mathbb{Z}_p$) and showed that this integral is related to non-archimedean combinatorial analysis in mathematical physics such as the functional equation of the q-zeta function, the q-Stirling numbers and q-Mahler theory, and so on. We refer the reader to [4, 5].

We now briefly summarize some properties of the usual Eulerian polynomials:

The Eulerian polynomials $\mathcal{A}_n(x)$ are defined as (known as the generating function of Eulerian poynomials)

$$e^{\mathcal{A}(x)t} = \sum_{n=0}^{\infty} \mathcal{A}_n(x)\frac{t^n}{n!} = \frac{1-x}{e^{t(1-x)}-x} \tag{1}$$

where we have used $\mathcal{A}^n(x) := \mathcal{A}_n(x)$, symbolically. The Eulerian polynomials can be generated via the recurrence relation:

$$(\mathcal{A}(t)+(t-1))^n - t\mathcal{A}_n(t) = \begin{cases} 1-t & \text{if } n=0 \\ 0 & \text{if } n\neq 0, \end{cases} \tag{2}$$

(for details, see [1]).

Suppose that p be a fixed odd prime number. Throughout this paper, we use the following notations. By $\mathbb{Z}_p$, we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{N}^* = \mathbb{N} \cup \{0\}$.

The normalized p-adic absolute value is defined by

S. Araci (✉)
Department of Economics, Faculty of Economics, Administrative and Social Science, Hasan Kalyoncu University, 27410 Gaziantep, Turkey
e-mail: mtsrkn@hotmail.com

M. Acikgoz · E. Ağyüz
Department of Mathematics, Faculty of Arts and Science, University of Gaziantep, 27310 Gaziantep, Turkey
e-mail: acikgoz@gantep.edu.tr

E. Ağyüz
e-mail: erkanagyuz@hotmail.com

$$|p|_p = \frac{1}{p}.$$

In this paper, we assume $|q-1|_p < 1$ as an indeterminate. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For a positive integer d with $(d,p)=1$, set

$$X = X_m = \varprojlim_{m} \mathbb{Z}/dp^m\mathbb{Z},$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p$$

and

$$a + dp^m\mathbb{Z}_p = \{x \in X \mid x \equiv a (\bmod dp^m)\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^m$.

Firstly, for introducing fermionic p-adic q-integral, we need some basic information which we state here. A measure on $\mathbb{Z}_p$ with values in a p-adic Banach space B is a continuous linear map

$$f \mapsto \int f(x)\mu = \int_{\mathbb{Z}_p} f(x)\mu(x)$$

from $C^0(\mathbb{Z}_p, \mathbb{C}_p)$, (continuous function on $\mathbb{Z}_p$) to B. We know that the set of locally constant functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ is dense in $C^0(\mathbb{Z}_p, \mathbb{C}_p)$ so.

Explicitly, for all $f \in C^0(\mathbb{Z}_p, \mathbb{C}_p)$, the locally constant functions

$$f_n = \sum_{i=0}^{p^m-1} f(i) 1_{i+p^m\mathbb{Z}_p} \to f \text{ in } C^0$$

Now, set $\mu(i+p^m\mathbb{Z}_p) = \int_{\mathbb{Z}_p} 1_{i+p^m\mathbb{Z}_p}\mu$. Then $\int_{\mathbb{Z}_p} f\mu$ is given by the following Riemann sum

$$\int_{\mathbb{Z}_p} f\mu = \lim_{m\to\infty} \sum_{i=0}^{p^m-1} f(i)\mu(i+p^m\mathbb{Z}_p)$$

The following q-Haar measure is defined by Kim in [2] and [4]:

$$\mu_q(a+p^m\mathbb{Z}_p) = \frac{q^a}{[p^m]_q}$$

So, for $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(\eta)d\mu_q(\eta) = \lim_{n\to\infty} \frac{1}{[p^n]_q} \sum_{\eta=0}^{p^n-1} q^\eta f(\eta). \tag{3}$$

The bosonic integral is considered as the bosonic limit $q \to 1$, $I_1(f) = \lim_{q\to 1} I_q(f)$. In [8, 9] and [10], similarly, the p-adic fermionic integration on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(f) = \lim_{q\to -q} I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-q}(x). \tag{4}$$

By (4), we have the following well-known integral equation:

$$q^n I_{-q}(f_n) + (-1)^{n-1} I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l f(l) \tag{5}$$

Here $f_n(x) := f(x+n)$. By (5), we have the following equalities:

If n odd, then

$$q^n I_{-q}(f_n) + I_{-q}(f) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l). \tag{6}$$

If n even, then we have

$$I_{-q}(f) - q^n I_{-q}(f_n) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l f(l). \tag{7}$$

Substituting $n=1$ into (6), we readily see the following

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0). \tag{8}$$

Replacing q by q^{-1} in (8), we easily derive the following

$$I_{-q^{-1}}(f_1) + qI_{-q^{-1}}(f) = [2]_q f(0). \tag{9}$$

In [1], Kim et al. considered $f(x) = e^{-x(1+q)t}$ in (9) and derived Witt's formula of the Eulerian polynomials as follows:

For $n \in \mathbb{N}^*$,

$$I_{-q^{-1}}(x^n) = \frac{(-1)^n}{(1+q)^n} \mathcal{A}_n(-q). \tag{10}$$

In the next section we will introduce $I_{-q^{-1}}(\chi(x)x^n)$ based on the fermionic p-adic q-integral in the p-adic integer ring which will be known as the Eulerian polynomials attached to χ (or Dirichlet's type of Eulerian polynomials) and we will give some new properties.

On the Dirichlet's type of Eulerian polynomials

Throughout this section, we always make use of d as an odd natural number. Firstly, we consider the following equality using (6):

$$\begin{aligned} &\int_{\mathbb{Z}_p} f(x+d)d\mu_{-q^{-1}}(x) + q^d \int_{\mathbb{Z}_p} f(x)d\mu_{-q^{-1}}(x) \\ &= [2]_q \sum_{0\leq l\leq d-1} (-1)^l q^{d-l+1} f(l). \end{aligned} \tag{11}$$

Let χ be a Dirichlet character of conductor d, which is any multiple of p (=*odd*). Then, substituting $f(x) = \chi(x)e^{-x(1+q)t}$ in (11), we compute as follows:

$$\int_{\mathbb{Z}_p} \chi(x+d)e^{-(x+d)(1+q)t}d\mu_{-q^{-1}}(x) + q^d \int_{\mathbb{Z}_p} \chi(x)e^{-x(1+q)t}d\mu_{-q^{-1}}(x) = [2]_q \sum_{0\leq l\leq d-1} (-1)^l q^{d-l+1}\chi(l)e^{-l(1+q)t}$$

After some applications, we see that

$$\int_{\mathbb{Z}_p} \chi(x)e^{-x(1+q)t}d\mu_{-q^{-1}}(x) = [2]_q \sum_{l=0}^{d-1}(-1)^l q^{d-l+1}\chi(l)\frac{e^{-l(1+q)t}}{e^{-d(1+q)t}+q^d}. \tag{12}$$

Let $\mathcal{F}_q(t \mid \chi) = \sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q)\frac{t^n}{n!}$. Then, we introduce the following definition of generating function of Dirichlet's type of Eulerian polynomials.

Definition 1 For $n \in \mathbb{N}^*$, we define the following:

$$\sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q)\frac{t^n}{n!} = [2]_q \sum_{l=0}^{d-1}(-1)^l q^{d-l+1}\chi(l)\frac{e^{-l(1+q)t}}{e^{-d(1+q)t}+q^d}. \tag{13}$$

By (12) and (13), we state the following theorem which is the Witt's formula for Dirichlet's type of Eulerian polynomials.

Theorem 2.1 The following identity holds true:

$$I_{-q^{-1}}(\chi(x)x^n) = \frac{(-1)^n}{(1+q)^n}\mathcal{A}_{n,\chi}(-q). \tag{14}$$

Using (13), we discover the following applications:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q)\frac{t^n}{n!} &= [2]_q \sum_{l=0}^{d-1}(-1)^l q^{d-l+1}\chi(l)\frac{e^{-l(1+q)t}}{e^{-d(1+q)t}+q^d} \\ &= [2]_q \sum_{l=0}^{d-1}(-1)^l q^{-l+1}\chi(l)e^{-l(1+q)t}\sum_{m=0}^{\infty}(-1)^m q^{-md}e^{-md(1+q)t} \\ &= q[2]_q \sum_{m=0}^{\infty}\sum_{l=0}^{d-1}(-1)^{l+md}\chi(l+md)q^{-(l+md)}e^{-(l+md)(1+q)t} \\ &= q[2]_q \sum_{m=0}^{\infty}(-1)^m \chi(m)q^{-m}e^{-m(1+q)t}.\end{aligned}$$

Thus, we get the following theorem.

Theorem 2.2 The following

$$\mathcal{F}_q(t \mid \chi) = \sum_{n=0}^{\infty} \mathcal{A}_{n,\chi}(-q)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}\frac{(-1)^m \chi(m)e^{-m(1+q)t}}{q^{m-1}} \tag{15}$$

is true.

By considering Taylor expansion of $e^{-m(1+q)t}$ in (15), we procure the following theorem.

Theorem 2.3 For $n \in \mathbb{N}$, we have

$$\frac{(-1)^n}{q(1+q)^{n+1}}\mathcal{A}_{n,\chi}(-q) = \sum_{m=1}^{\infty}\frac{(-1)^m \chi(m)m^n}{q^m}. \tag{16}$$

From (14) and (16), we easily derive the following corollary:

Corollary 2.4 For $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty}\sum_{m=1}^{p^n-1}\frac{(-1)^m \chi(m)m^n}{q^{m+1}} = 2\sum_{m=1}^{\infty}\frac{(-1)^m \chi(m)m^n}{q^{m-1}}.$$

We now give distribution formula for Dirichlet's type of Eulerian polynomials using p-adic q-integral on $\mathbb{Z}_p$, as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} \chi(x)x^n d\mu_{-q^{-1}}(x) &= \lim_{m\to\infty}\frac{1}{[dp^m]_{-q^{-1}}}\sum_{x=0}^{dp^m-1}(-1)^x \chi(x)x^n q^{-x} \\ &= \frac{d^n}{[d]_{-q^{-1}}}\sum_{a=0}^{d-1}(-1)^a \chi(a)q^{-a} \\ &\quad \times \left(\lim_{m\to\infty}\frac{1}{[p^m]_{-q^{-d}}}\sum_{x=0}^{p^m-1}(-1)^x\left(\frac{a}{d}+x\right)^n q^{-dx}\right) \\ &= \frac{d^n}{[d]_{-q^{-1}}}\sum_{a=0}^{d-1}(-1)^a \chi(a)q^{-a}\int_{\mathbb{Z}_p}\left(\frac{a}{d}+x\right)^n d\mu_{-q^{-d}}(x).\end{aligned}$$

Thus, we state the following theorem.

Theorem 2.5 The following identity holds true:

$$\frac{(-1)^n}{(1+q)^n}\mathcal{A}_{n,\chi}(-q) = \frac{d^n}{[d]_{-q^{-1}}}\sum_{a=0}^{d-1}(-1)^a \chi(a)q^{-a} \times \int_{\mathbb{Z}_p}\left(\frac{a}{d}+x\right)^n d\mu_{-q^{-d}}(x). \tag{17}$$

Notice that the Eq. (17) is related to q-Genocchi polynomials with weight zero, $\widetilde{G}_{n,q}(x)$, and q-Euler polynomials with weight zero, $\widetilde{E}_{n,q}(x)$, which is defined by Araci et al. and Kim and Choi in [21] and [11], respectively, as follows:

$$\frac{\widetilde{G}_{n+1,q}(x)}{n+1} = \lim_{m\to\infty}\frac{1}{[p^m]_{-q}}\sum_{y=0}^{p^m-1}(-1)^y(x+y)^n q^y \tag{18}$$

and

$$\widetilde{E}_{n,q}(x) = \int_{\mathbb{Z}_p}(x+y)^n d\mu_{-q}(y). \tag{19}$$

By expressions of (17), (18) and (19), we easily discover the following corollary.

Corollary 2.6 For $n \in \mathbb{N}^*$, we have

$$\frac{(-1)^n}{(1+q)^n}\mathcal{A}_{n,\chi}(-q) = \frac{d^n}{(n+1)[d]_{-q^{-1}}}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^{-a}\widetilde{G}_{n+1,q^{-d}}\left(\frac{a}{d}\right).$$

Moreover,

$$\frac{(-1)^n}{(1+q)^n}\mathcal{A}_{n,\chi}(-q) = \frac{d^n}{[d]_{-q^{-1}}}\sum_{a=0}^{d-1}(-1)^a\chi(a)q^{-a}\widetilde{E}_{n,q^{-d}}\left(\frac{a}{d}\right).$$

On the Eulerian L type function

Our goal in this section is to introduce Eulerian L type function by applying Mellin transformation to the generating function of Dirichlet's type of Eulerian polynomials. By (15), for $s \in \mathbb{C}$, we define the following

$$L_E(s \mid \chi) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathcal{F}_q(t \mid \chi)dt$$

where $\Gamma(s)$ is the Euler Gamma function. It becomes as follows:

$$\begin{aligned} L_E(s \mid \chi) &= q[2]_q\sum_{m=0}^{\infty}(-1)^m\chi(m)q^{-m}\left\{\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-m(1+q)t}dt\right\} \\ &= \frac{q}{(1+q)^{s-1}}\sum_{m=1}^{\infty}\frac{(-1)^m\chi(m)}{q^m m^s} \end{aligned}$$

So, we give definition of Eulerian L type function as follows:

Definition 2 For $s \in \mathbb{C}$, then we have

$$L_E(s \mid \chi) = \frac{q}{(1+q)^{s-1}}\sum_{m=1}^{\infty}\frac{(-1)^m\chi(m)}{q^m m^s}. \tag{20}$$

Substituting $s = -n$ into (20) and comparing with the Eq. (16), then, relation between Eulerian L type function and Dirichlet's type of Eulerian polynomials is given by the following theorem.

Theorem 3.1 The following equality holds true:

$$L_E(-n \mid \chi) = \begin{cases} -\mathcal{A}_{n,\chi}(-q) & \text{if n odd,} \\ \mathcal{A}_{n,\chi}(-q) & \text{if n even.} \end{cases}$$

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