

Research Article

Results Concerning the Analysis of Multi-Index Whittaker Function

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A variety of functions, their extensions, and variants have been extensively investigated, mainly due to their potential applications in diverse research areas. In this paper, we aim to introduce a new extension of Whittaker function in terms of multi-index confluent hypergeometric function of first kind. We discuss multifarious properties of newly defined multi-index Whittaker function such as integral representation, integral transform (i.e., Mellin transform and Hankel transform), and derivative formula. The results presented here, being very general, are pointed out to reduce to yield some known or new formulas and identities for relatively functions.

1. Introduction

Generalized and multivariable forms of the special functions of mathematical physics have witnessed a significant evolution during recent years. In particular, the special functions of more than one variable provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special functions of mathematical physics and their generalization have been suggested by physical problems. In mathematics, the Whittaker function is a solution of Whittaker equation, which is a modified form of confluent hypergeometric function of first kind, and it has various applications in multifarious area such as mathematical physics and many research areas, which are studied by various mathematicians (see [1–4]). Recently, many authors give an extension and generalization of several special functions such as beta function, gamma function, hypergeometric function, confluent hypergeometric function, and Whittaker function (see [2, 3, 5–12]). Ghayasuddin et al. [7] defined a new type of confluent hypergeometric function by

using extended beta function in terms of multi-index Mittag–Leffler function. Inspired by the abovementioned work, in this paper, we introduced a new extension of Whittaker function in terms of multi-index Mittag–Leffler function by using an extended confluent hypergeometric function and studied their various properties such as integral transform, integral representation, and derivative formula of it. We remember below the following basic definition and extension of special function.

The classical beta function $B(u, v)$ is defined as (see [13])

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad (1)$$

where

$$(\Re(u) > 0, \Re(v) > 0). \quad (2)$$

The classical Gauss hypergeometric function $F(u, v; w; \omega)$ and confluent hypergeometric function $\Phi(v; w; \omega)$ are defined as (see [14])

$$F(u, v; w; \omega) = \frac{1}{B(v, w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} (1-\omega t)^{-u} dt, \quad (|\arg(1-\omega)| < \pi; \Re(w) > \Re(v) > 0), \quad (3)$$

$$\Phi(v; w; \omega) = \frac{1}{B(v, w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} \exp(\omega t) dt, \quad (\Re(w) > \Re(v) > 0). \quad (4)$$

By using the series expansion of $(1-\omega t)^{-u}$ and $\exp(\omega t)$ in (3) and (4), respectively, the hypergeometric and

confluent hypergeometric functions are written in terms of beta function as

$$F(u, v; w; \omega) = \sum_{n=0}^{\infty} (u)_n \frac{B(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (|\omega| < 1, \Re(w) > \Re(v) > 0), \quad (5)$$

$$\Phi(v; w; \omega) = \sum_{n=0}^{\infty} \frac{B(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (\Re(w) > \Re(v) > 0).$$

In 1997, Aslam Chaudhary et al. [5] give an extension of beta function defined as

$$B_\rho(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} \exp\left(-\frac{\rho}{t(1-t)}\right) dt, \quad (6)$$

where

$$(\Re(\rho) > 0, \Re(u) > 0, \Re(v) > 0). \quad (7)$$

Remark 1. If $\rho = 0$, then extended beta function (6) is reduced to classical beta function (1).

In 2004, Chaudhary et al. [6] introduced the extended hypergeometric and confluent hypergeometric functions in terms of extended beta function (6) as follows:

$$F_\rho(u, v; w; \omega) = \sum_{n=0}^{\infty} (u)_n \frac{B_\rho(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (\rho \geq 0, |\omega| < 1, \Re(w) > \Re(v) > 0), \quad (8)$$

$$\Phi_\rho(v; w; \omega) = \sum_{n=0}^{\infty} \frac{B_\rho(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (\rho \geq 0, \Re(w) > \Re(v) > 0).$$

Their integral representation is

$$F_\rho(u, v; w; \omega) = \frac{1}{B(v, w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} (1-\omega t)^{-u} \exp\left(-\frac{\rho}{t(1-t)}\right) dt, \quad (\rho > 0; \rho = 0 \text{ and } |\arg(1-\omega)| < \pi; \Re(w) > \Re(v) > 0), \quad (9)$$

$$\Phi_\rho(v; w; \omega) = \frac{1}{B(v, w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} \exp\left(\omega t - \frac{\rho}{t(1-t)}\right) dt,$$

$$(\rho > 0; \rho = 0 \text{ and } \Re(w) > \Re(v) > 0).$$

Shadab et al. [12] introduced an extension of beta function using generalized Mittag–Leffler function as follows:

$$B_{\alpha}^{\rho}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} E_{\alpha}\left(-\frac{\rho}{t(1-t)}\right) dt, \tag{10}$$

$$(\Re(u) > 0, \Re(v) > 0, \Re(\rho) > 0; \alpha \in \mathbb{R}_0^+),$$

where $E_{\alpha}(\cdot)$ is the classical Mittag–Leffler (see [15, 16]) function defined by

$$E_{\alpha}(\omega) = \sum_{n=0}^{\infty} \frac{\omega^n}{\Gamma(\alpha n + 1)}, \tag{11}$$

where

$$(\omega \in \mathbb{C}, \alpha \in \mathbb{R}_0^+). \tag{12}$$

Shadab et al. [12] expressed the extended hypergeometric and confluent hypergeometric functions in terms of extended beta function (11) as follows:

$$F_{\rho, \alpha}(u, v; w; \omega) = \sum_{n=0}^{\infty} (u)_n \frac{B_{\alpha}^{\rho}(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (\alpha \in \mathbb{R}^+, \rho \in \mathbb{R}_0^+, |\omega| < 1, \Re(w) > \Re(v) > 0), \tag{13}$$

$$\Phi_{\rho, \alpha}(v; w; \omega) = \sum_{n=0}^{\infty} \frac{B_{\alpha}^{\rho}(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (\alpha \in \mathbb{R}^+, \rho \in \mathbb{R}_0^+, |\omega| < 1, \Re(w) > \Re(v) > 0),$$

and their integral representation is

$$F_{\rho, \alpha}(u, v; w; \omega) = \frac{1}{B(v, w-v)} \times \int_0^1 t^{v-1} (1-t)^{w-v-1} (1-\omega t)^{-u} E_{\alpha}\left(-\frac{\rho}{t(1-t)}\right) dt, \tag{14}$$

$$(\alpha \in \mathbb{R}^+, \rho > 0; \rho = 0 \text{ and } |\arg(1-\omega)| < \pi; \Re(w) > \Re(v) > 0),$$

$$\Phi_{\rho, \alpha}(v; w; \omega) = \frac{1}{B(v, w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} e^{\omega t} E_{\alpha}\left(-\frac{\rho}{t(1-t)}\right) dt,$$

$$(\alpha \in \mathbb{R}^+, \rho > 0; \rho = 0 \text{ and } \Re(w) > \Re(v) > 0).$$

Ghayasuddin et al. [7] introduced an extension of beta function using multi-index Mittag–Leffler function as follows:

$$B_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} E_{(1/a_i)(b_i)}\left(-\frac{\rho}{t(1-t)}\right) dt, \quad (\Re(u) > 0, \Re(v) > 0, \Re(\rho) > 0; a_i > 0, b_i \in \mathbb{R}), \tag{15}$$

For $s=2$, if we set $1/a_1 = \alpha, 1/a_2 = 0$, and $b_1 = b_2 = 1$ in (15), then we obtain the extended beta function defined by Shadab et al. [12].

Multi-index Mittag–Leffler function $E_{(1/a_i)(b_i)}(\cdot)$ is defined as follows (see [17]):

$$E_{(1/a_i)(b_i)}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{\Gamma(b_1 + (k/a_1)) \dots \Gamma(b_s + (k/a_s))}. \tag{16}$$

where $s > 1$ is an integer and $a_1, a_2, \dots, a_s > 0$ and b_1, b_2, \dots, b_s are arbitrary real numbers.

It is easy to see that, for $s = 2, (1/a_1) = \alpha, (1/a_2) = 0$, and $b_1 = b_2 = 1$, then multi-index Mittag–Leffler function is reduced to classical Mittag–Leffler function $E_{\alpha}(\omega)$.

In 2020, Ghayasuddin et al. [7] expressed the extended hypergeometric and confluent hypergeometric functions in terms of extended beta function (15) as follows:

$$F_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(u, v; w; \omega) = \sum_{n=0}^{\infty} (u)_n \frac{B_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (a_i > 0, b_i \in \mathbb{R}, \rho \in \mathbb{R}_0^+, |\omega| < 1, \Re(w) > \Re(v) > 0), \tag{17}$$

$$\Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(v; w; \omega) = \sum_{n=0}^{\infty} \frac{B_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(v+n, w-v)}{B(v, w-v)} \frac{\omega^n}{n!}, \quad (a_i > 0, b_i \in \mathbb{R}, \rho \in \mathbb{R}_0^+, |\omega| < 1, \Re(w) > \Re(v) > 0),$$

and their integral representation is

$$F_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(u, v; w; \omega) = \frac{1}{B(v, w-v)} \int_0^1 t^{v-1} (1-t)^{w-v-1} (1-\omega t)^{-u} E_{(1/a_i)(b_i)}\left(-\frac{\rho}{t(1-t)}\right) dt, \tag{18}$$

$$(\rho > 0; \rho = 0 \text{ and } |\arg(1-\omega)| < \pi; \Re(w) > \Re(v) > 0, a_i > 0, b_i \in \mathbb{R}),$$

$$\Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(v; w; \omega) = \frac{1}{B(v, w-v)} \times \int_0^1 t^{v-1} (1-t)^{w-v-1} e^{\omega t} E_{(1/a_i)(b_i)}\left(-\frac{\rho}{t(1-t)}\right) dt, \tag{19}$$

$$(\rho > 0; \rho = 0 \text{ and } \Re(w) > \Re(v) > 0, a_i > 0, b_i \in \mathbb{R}).$$

The extension of Kummer’s relation to the generalized extended confluent hypergeometric function of the first kind is as follows:

$$\Phi_{\alpha, \beta}^{\rho, \eta, \nu}(v, w; \omega) = e^{\omega} \Phi_{\alpha, \beta}^{\rho, \eta, \nu}(w-v; w; -\omega). \tag{20}$$

For $\alpha = \beta = \eta = \nu = 1$ and $\rho = 0$, (20) is reduced to Kummer’s formula of first kind for the classical confluent hypergeometric function (see [14]).

The Whittaker function $M_{\kappa, \zeta}(\omega)$ in terms of confluent hypergeometric function of first kind (see [4, 18]) is defined as

$$M_{\kappa, \zeta}(\omega) = \omega^{\zeta+(1/2)} \exp\left(\frac{-\omega}{2}\right) \Phi\left(\zeta - \kappa + \frac{1}{2}; 2\zeta + 1; \omega\right), \quad \left(\Re(\zeta) > \frac{-1}{2} \text{ and } \Re(\zeta \pm \kappa) > \frac{-1}{2}\right). \tag{21}$$

In 2013, Nagar et al. [3] generalized the Whittaker function by using extended confluent hypergeometric function Φ_{ρ} which is defined as

$$M_{\rho, \kappa, \zeta}(\omega) = \omega^{\zeta+(1/2)} \exp\left(\frac{-\omega}{2}\right) \Phi_{\rho}\left(\zeta - \kappa + \frac{1}{2}; 2\zeta + 1; \omega\right), \quad \left(\Re(\zeta) > \frac{-1}{2} \text{ and } \Re(\zeta \pm \kappa) > \frac{-1}{2}\right). \tag{22}$$

2. Multi-Index Whittaker Function

In this section, we give a new generalization of Whittaker function of the first kind by applying the multi-index confluent hypergeometric function (19) defined as

$$M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) = \omega^{\zeta+1/2} \exp\left(\frac{-\omega}{2}\right) \Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}\left(\zeta - \kappa + \frac{1}{2}; 2\zeta + 1; \omega\right), \quad \left(\rho \geq 0, \Re(\zeta) > \frac{-1}{2} \text{ and } \Re(\zeta \pm \kappa) > \frac{-1}{2}, a_i > 0, b_i \in \mathbb{R}\right). \tag{23}$$

If we take $\rho = 0$ and $a_1 = \dots = a_s = b_1 = \dots = b_s = 1$, (23) reduced to the classical Whittaker function (21).

2.1. *Integral Representation.* Here, we define integral representation of the multi-index Whittaker function by using (19) and (23) as

$$M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) = \frac{\omega^{\zeta+1/2} \exp(-\omega/2)}{B(\zeta - \kappa + 1/2, \zeta + \kappa + 1/2)} \times \int_0^1 t^{\zeta - \kappa - 1/2} (1-t)^{\zeta + \kappa - 1/2} e^{\omega t} E_{(1/a_i)}(b_i) \left(-\frac{\rho}{t(1-t)} \right) dt$$

$$\cdot \left(\rho \geq 0, \Re(\zeta) > \frac{-1}{2} \text{ and } \Re(\zeta \pm \kappa) > \frac{-1}{2}, a_i > 0, b_i \in \mathbb{R} \right).$$

Substituting $t = (x - a)/(b - a)$ in (24), we get the multi-index Whittaker function as

$$M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) = \frac{(b-a)^{-2\zeta+1} \omega^{\zeta+(1/2)} \exp(-\omega/2)}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))}$$

$$\times \int_a^b (x-a)^{\zeta - \kappa - (1/2)} (b-x)^{\zeta + \kappa - (1/2)} e^{\omega(x-a/b-a)} E_{(1/a_i)}(b_i) \left(-\frac{\rho(b-a)^2}{(x-a)(b-x)} \right) dx,$$

where a and b are scalars such that $(b - a) > 0$.

If we take $a = -1$ and $b = 1$ in (23), we obtain another representation of multi-index Whittaker function:

$$M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) = \frac{2^{-2\zeta+1} \omega^{\zeta+(1/2)}}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \times \int_a^b (x+1)^{\zeta - \kappa - (1/2)} (1-x)^{\zeta + \kappa - (1/2)} e^{\omega x/2} E_{(1/a_i)}(b_i) \left(-\frac{4\rho}{(x+1)(1-x)} \right) dx,$$

In (24), substitute $t = x/(1+x)$, and we get another integral representation of multi-index Whittaker function as

$$M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) = \frac{\omega^{\zeta+(1/2)} \exp(-\omega/2)}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))}$$

$$\times \int_0^\infty x^{\zeta - \kappa - (1/2)} (1+x)^{-(2\zeta+1)} e^{\omega x/(1+x)} E_{(1/a_i)}(b_i) \left(-\frac{\rho(1+x)^2}{x} \right) dx.$$

If $a_1 = \dots = a_s = b_1 = \dots = b_s = 1$ and $\rho = 0$ in (24), (23), and (27), we obtain integral representation of classical Whittaker function.

$$M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(-\omega) = (-1)^{\zeta+(1/2)} M_{\rho, -\eta, \nu}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega), \quad (\rho \geq 0).$$

Theorem 1. *The following relation holds true:*

Proof. Using relation (20) in (23), we have

$$M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(-\omega) = (-1)^{\zeta+(1/2)} \omega^{\zeta+(1/2)} \exp\left(\frac{-\omega}{2}\right) \Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}\left(\zeta - \kappa + \frac{1}{2}; 2\zeta + 1; \omega\right).$$

Now, writing the right-hand side of the above representation by using (23), we get the desired result. \square

3. Integral Transform of Multi-Index Whittaker Function

Theorem 2. *The following Mellin transform formula holds true:*

$$\begin{aligned} & \int_0^\infty \rho^{s-1} M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) d\rho \\ &= \frac{\Gamma(s)\Gamma(1-s)}{\prod_{i=1}^m \Gamma(b_i - (s/a_i))} \frac{B((\zeta + s) - \kappa + (1/2), 2(\zeta + s) + 1)}{\omega^s B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} M_{\kappa, \zeta + s}(\omega) \\ & \cdot \left(\Re(s) > 0 \text{ and } \Re(\zeta \pm \kappa) > \frac{-1}{2}, a_i > 0, b_i \in \mathbb{R} \right). \end{aligned} \quad (30)$$

Proof. Using multi-index Whittaker function (1), we obtain

$$\int_0^\infty \rho^{s-1} M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) d\rho = \omega^{\zeta + (1/2)} \exp\left(\frac{-\omega}{2}\right) \int_0^\infty \rho^{s-1} \Phi_\rho^{a_1, \dots, a_s, b_1, \dots, b_s}\left(\zeta - \kappa + \frac{1}{2}; 2\zeta + 1; \omega\right) d\rho. \quad (31)$$

Again, using multi-index confluent hypergeometric function (19), we obtain

$$= \frac{\omega^{\zeta + (1/2)} \exp(-\omega/2)}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \times \int_0^\infty \rho^{s-1} \int_0^1 t^{\zeta - \kappa - (1/2)} (1-t)^{\zeta + \kappa - (1/2)} e^{\omega t} E_{(1/a_i)}(b_i)\left(\frac{\rho}{t(1-t)}\right) dt d\rho. \quad (32)$$

Changing the order of integration, we obtain

$$= \frac{\omega^{\zeta + (1/2)} \exp(-\omega/2)}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \times \int_0^1 t^{\zeta - \kappa - (1/2)} (1-t)^{\zeta + \kappa - (1/2)} e^{\omega t} \int_0^\infty \rho^{s-1} E_{(1/a_i)}(b_i)\left(\frac{\rho}{t(1-t)}\right) d\rho dt. \quad (33)$$

Substituting $u = \rho/(t(1-t))$ in integral (33), we obtain

$$\begin{aligned} &= \frac{\omega^{\zeta + (1/2)} \exp(-\omega/2)}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \\ & \times \int_0^1 t^{\zeta + s - \kappa - (1/2)} (1-t)^{\zeta + \kappa - (1/2)} e^{\omega t} \int_0^\infty u^{s-1} E_{(1/a_i)}(b_i)(-u) du dt. \end{aligned} \quad (34)$$

Now, using well-known result (p. 102 of [19]) and confluent hypergeometric function (4), we obtain

$$\begin{aligned}
 &= \frac{\omega^{\zeta+(1/2)} \exp(-\omega/2) B((\zeta + s) - \kappa + (1/2), 2(\zeta + s) + 1)}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \\
 &\quad \times \phi(\zeta + s - \kappa + (1/2); 2(\zeta + s) + 1; \omega) \frac{\Gamma(s)\Gamma(1-s)}{\prod_{i=0}^m \Gamma(b_i - (s/a_i))} \\
 &= \frac{\omega^{\zeta+s+(1/2)} \exp(-\omega/2) \phi((\zeta + s) - \kappa + (1/2); 2(\zeta + s) + 1; \omega)}{\omega^s B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \\
 &\quad \times B\left((\zeta + s) - \kappa + \frac{1}{2}, 2(\zeta + s) + 1\right) \frac{\Gamma(s)\Gamma(1-s)}{\prod_{i=0}^m \Gamma(b_i - (s/a_i))}.
 \end{aligned} \tag{35}$$

By using definition of classical Whittaker function (21), we get the desired result. \square

Theorem 3. *The following formula holds true:*

$$\begin{aligned}
 &\int_0^\infty \omega^{a-1} e^{-b\omega} M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(c\omega) d\omega \\
 &= \frac{c^{\zeta+(1/2)} \Gamma(a + \zeta + (1/2))}{(b + (c/2))^{a+\zeta+(1/2)}} F_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s} \left(a + \zeta + \frac{1}{2}, \zeta - \kappa + \frac{1}{2}, 2\zeta + 1; \frac{2c}{2b + c} \right) \\
 &\quad \cdot \left(\rho \geq 0, 2b > c > 0, \Re(a + \zeta) > \frac{-1}{2}, a_i > 0, b_i \in \mathbb{R} \right).
 \end{aligned} \tag{36}$$

Proof. Using integral representation of the multi-index Whittaker function, we obtain

$$\begin{aligned}
 &\int_0^\infty \omega^{a-1} e^{-b\omega} M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(c\omega) d\omega \\
 &= \frac{1}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \int_0^\infty \omega^{a-1} e^{-b\omega} (c\omega)^{\zeta+(1/2)} \exp\left(\frac{-c\omega}{2}\right) \\
 &\quad \times \int_0^1 t^{\zeta - \kappa - (1/2)} (1-t)^{\zeta + \kappa - (1/2)} e^{c\omega t} E_{(1/a_i)}(b_i) \left(\frac{\rho}{t(1-t)} \right) dt d\omega.
 \end{aligned} \tag{37}$$

Now, interchanging the order of integration and using the definition of gamma function, we obtain

$$\begin{aligned}
 &\int_0^\infty \omega^{a-1} e^{-b\omega} M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(c\omega) d\omega \\
 &= \frac{c^{\zeta+(1/2)} \Gamma(a + \zeta + (1/2))}{(b + (c/2))^{a+\zeta+(1/2)} B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2))} \\
 &\quad \times \int_0^1 t^{\zeta - \kappa - (1/2)} (1-t)^{\zeta + \kappa - (1/2)} \left(1 - \frac{2ct}{2b + c} \right)^{-(a+\zeta+(1/2))} E_{(1/a_i)}(b_i) \left(\frac{\rho}{t(1-t)} \right) dt d\omega.
 \end{aligned} \tag{38}$$

Using (18) in (38), we get the desired result (36). \square

Corollary 1. *If we take $a = c = 1$ in (36), we get the following special cases:*

$$\int_0^\infty e^{-b\omega} M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) d\omega = \frac{2^{\zeta+(3/2)} \Gamma(\zeta + (3/2))}{(2b + 1)^{\zeta+(3/2)}} F_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s} \left(\zeta + \frac{3}{2}, \zeta - \kappa + \frac{1}{2}; 2\zeta + 1; \frac{2}{2b + c} \right). \tag{39}$$

Theorem 4. *The following Hankel transform formula holds true:*

$$\int_0^\infty \omega M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) J_m(a\omega) d\omega = \frac{\Gamma(\zeta + \kappa + (5/2))}{(a^2 + (1/4))^{(\zeta/2)+(5/4)}} \sum_{n=0}^\infty \frac{B_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(\zeta - \kappa + n + (1/2), \zeta + \kappa + (1/2)) (\zeta + m + (5/2))_n}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2)) (a^2 + (1/4))^{n/2} n!} \times P_{\zeta+n+(3/2)}^{-m} \left(\frac{1}{\sqrt{4a^2 + 1}} \right), \tag{40}$$

$$\left(\Re(\zeta \pm \kappa) > \frac{-1}{2} \text{ and } \Re(\zeta + m) > \frac{-5}{2}, a_i > 0, b_i \in \mathbb{R} \right), \tag{41}$$

where $P_{\zeta}^m(z)$ is the Legendre function (see p.34. of [20]).

Proof. Using (23) and (18), expanding multi-index Whittaker function in terms of generating extended beta function and changing the order of integration and summation, we obtain

$$\int_0^\infty z M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) J_m(a\omega) d\omega = \sum_{n=0}^\infty \frac{B_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s}(\zeta - \kappa + n + (1/2), \zeta + \kappa + (1/2))}{B(\zeta - \kappa + (1/2), \zeta + \kappa + (1/2)) n!} \times \int_0^\infty \omega^{\zeta+n+(3/2)} e^{-\omega/2} J_m(a\omega) d\omega. \tag{42}$$

On using the known result (see p. 182 (9) of [21]),

$$\int_0^\infty e^{-pt} t^{\zeta} J_m(at) dt = \Gamma(\zeta + m + 1) r^{-\zeta-1} P_{\zeta}^{-m} \left(\frac{p}{r} \right), \tag{43}$$

where $\Re(\zeta + m) > -1, r = (p^2 + a^2)^{1/2}$ and $P_{\zeta}^m(z)$ is the Legendre function (see [20]). By using (43) in (42) and after some simplification, we get the desired result. \square

4. Derivative of Multi-Index Whittaker Function

Theorem 5. *The following differential formula holds true:*

$$\frac{d^n}{d\omega^n} \left[e^{\omega/2} \omega^{-\zeta-(1/2)} M_{\rho, \kappa, \zeta}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega) \right] = \frac{(\zeta - \kappa + (1/2))_n}{(2\zeta + 1)_n} e^{\omega/2} \omega^{-\zeta-(n/2)-(1/2)} M_{\rho, \kappa}^{-\frac{n}{2}, \zeta + \frac{n}{2}}^{a_1, \dots, a_s, b_1, \dots, b_s}(\omega). \tag{44}$$

Proof. The n^{th} order derivative of generalized extended confluent hypergeometric function is given by [9]

$$\begin{aligned} & \frac{d^n}{d\omega^n} \left[\Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s} (v; \omega) \right] \\ &= \frac{(v)_n}{(\omega)_n} \Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s} (v+n; \omega+n; \omega). \end{aligned} \tag{45}$$

Now, using (23) on the left-hand side of (44), we obtain

$$\begin{aligned} & \frac{d^n}{d\omega^n} \left[e^{\omega/2} z^{-\zeta-(1/2)a_1, \dots, a_s, b_1, \dots, b_s} \Big|_{\rho, \kappa, \zeta} (\omega) \right] \\ &= \frac{d^n}{d\omega^n} \left[\Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s} \left(\zeta - \kappa + \frac{1}{2}; 2\zeta + 1; \omega \right) \right] \\ &= \frac{(\zeta - \kappa + (1/2))_n}{(2\zeta + 1)_n} \times \Phi_{\rho}^{a_1, \dots, a_s, b_1, \dots, b_s} \left[\left(\zeta + \frac{n}{2} \right) - \left(\kappa - \frac{n}{2} \right) + \frac{1}{2}; 2 \left(\zeta + \frac{1}{2} \right) + 1; \omega \right] \\ &= \frac{(\zeta - \kappa + (1/2))_n}{(2\zeta + 1)_n} e^{\omega/2} \omega^{-\zeta-(n/2)-(1/2)} M_{\rho, \kappa-(n/2), \zeta+(n/2)}^{a_1, \dots, a_s, b_1, \dots, b_s} (\omega). \end{aligned} \tag{46}$$

We get the desired result (44). □

5. Conclusion

In the present paper, we introduce a multi-index Whittaker function in terms of extended confluent hypergeometric function. We have provided some important properties of Whittaker function such as integral representation, integral transform, and derivative formula. We have known that most of the special function of mathematical physics such as modified Bessel function and Laguerre and Hermite polynomial can be written in terms of Whittaker function. Therefore, extensions and generalization of the Whittaker function are playing important roles in applied mathematics and mathematical physics.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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