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Symmetric functions of the *k*-Fibonacci and *k*-Lucas numbers

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In this paper, we introduce a new operator in order to derive some new symmetric properties of k-Fibonacci and k-Lucas numbers and Fibonacci polynomials. By making use of the new operator defined in this paper, we give some new generating functions for k-Fibonacci and Pell numbers and Fibonacci polynomials.

Keywords: k-Fibonacci numbers; k-Lucas numbers; generating functions; Fibonacci polynomials.

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1. Introduction and Notations

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art (e.g. see [27]). Fibonacci numbers F_n are defined by the recurrence relation

$$\begin{cases} F_0 = 1, \quad F_1 = 1, \\ F_{n+1} = F_n + F_{n-1}, \quad n \geq 1. \end{cases}$$

There exist a lot of properties about Fibonacci numbers. In particular, there is a beautiful combinatorial identity to Fibonacci numbers [27].

$$F_n = \sum_{i=0}^{\left[\frac{n-1}{2}\right]} \binom{n-i-1}{i}. \tag{1.1}$$

From (1.1), Filipponi [22] introduced the incomplete Fibonacci numbers $F_n(s)$ and the incomplete Lucas numbers $L_n(s)$. They are defined by

$$F_n(s) = \sum_{j=0}^{s} \binom{n-j-1}{j}, \left(0 \leq s \leq \left[\frac{n-1}{2}\right];\ n = 0, 1, 2, \ldots\right),$$

$$L_n(s) = \sum_{j=0}^{s} \binom{n-j}{j}, \left(0 \leq s \leq \left[\frac{n}{2}\right];\ n = 0, 1, 2, \ldots\right).$$

In [17], Djordjevic gave the incomplete generalized Fibonacci and Lucas numbers. In [18], Djordjevic and Srivastava defined incomplete generalized Jacobsthal and Jacobsthal–Lucas numbers. In [16], the authors defined the incomplete Fibonacci and Lucas numbers. For the systematic work related to Fibonacci and related numbers, we refer the readers to see the references [29–32].

On the other hand, many kinds of generalizations of Fibonacci numbers have been presented in the literature. In particular, a generalization is the k-Fibonacci numbers. For any positive real number k, the k-Fibonacci sequence, say $(F_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by

$$\begin{cases} F_{k,0} = 1, \quad F_{k,1} = 1, \\ F_{k,n+1} = kF_{k,n} + F_{k,n-1}, \quad n \geq 1. \end{cases}$$

In [24], k-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [24, 25].

For any positive real number k, the k-Lucas sequence, say $(L_{n,k})_{n\in\mathbb{N}}$, is defined recurrently by

$$\begin{cases} L_{k,0} = 2, \quad L_{k,1} = k, \\ L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \quad n \geq 1. \end{cases}$$

If $k = 2$, we have the classical Pell numbers appears: $P_0 = 0, P_1 = 1$ and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 1$, *cf.* [3].

In this contribution, we shall define a new useful operator denoted by $\delta_{e_1e_2}^{-k}$ for which we can formulate, extend and prove new results based on our previous ones [3, 5]. In order to determine new generating functions of the products of some known numbers and polynomails, we combine between our indicated past techniques and these presented polishing approaches.

Let k and n be two positive integer and $\{a_1, a_2, \ldots, a_n\}$ are set of given variables, recall [20] that the kth elementary symmetric function $e_k(a_1, a_2, \ldots, a_n)$ and the kth complete homogeneous symmetric function $h_k(a_1, a_2, \ldots, a_n)$ are defined respectively by

$$e_k(a_1, a_2, \ldots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, \ldots, i_n = 0$ or 1,

$$h_k(a_1, a_2, \ldots, a_n) = \sum_{i_1+i_2+\cdots+i_n=k} a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}, \quad 0 \leq k \leq n,$$

with $i_1, i_2, \ldots, i_n \geq 0$.

First, we set $e_0(a_1, a_2, \ldots, a_n) = 1$ and $h_0(a_1, a_2, \ldots, a_n) = 1$ (by convention). For $k > n$ or $k < 0$, we set $e_k(a_1, a_2, \ldots, a_n) = 0$ and $h_k(a_1, a_2, \ldots, a_n) = 0$.

Definition 1.1. Let $E = \{e_1, e_2\}$ an alphabet, we define the symmetric function S_n associated with the alphabet E by

$$S_n(e_1 + e_2) = h_n(e_1, e_2) = \frac{e_1^{n+1} - e_2^{n+1}}{e_1 - e_2}, \tag{1.2}$$

with

$$\begin{aligned} S_0(e_1 + e_2) &= h_0(e_1, e_2) = 1, \\ S_1(e_1 + e_2) &= h_1(e_1, e_2) = e_1 + e_2, \\ S_2(e_1 + e_2) &= h_2(e_1, e_2) = e_1^2 + e_1 e_2 + e_2^2, \\ &\vdots \end{aligned}$$

Definition 1.2. Let A and B be any two alphabets, then we give $S_n(A - B)$ by the following form:

$$\frac{\Pi_{b\in B}(1 - bz)}{\Pi_{a\in A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - B)z^n \tag{1.3}$$

with the condition $S_n(A - B) = 0$ for $n < 0$ (see [1]).

Corollary 1.1. *Taking* $A = 0$ *in* (1.3) *gives*

$$\Pi_{b\in B}(1 - bz) = \sum_{n=0}^{\infty} S_n(-B)z^n \tag{1.4}$$

Further, in the case $A = 0$ *or* $B = 0$, *we have*

$$\sum_{n=0}^{\infty} S_n(A - B)z^n = \sum_{n=0}^{\infty} S_n(A)z^n \sum_{n=0}^{\infty} S_n(-B)z^n.$$

Definition 1.3. Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \ldots, x_i, x_{i+1}, \ldots x_n) - g(x_1, \ldots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}}$$

(see [19]).

Definition 1.4 ([9]). The symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by

$$\delta^k_{e_1 e_2}(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \quad (k \in \mathbb{N}). \tag{1.5}$$

Remark 1.1. Let $E = \{e_1, e_2\}$ an alphabet, we have

$$h_k(e_1, e_2) = S_k(e_1 + e_2) = \delta^k_{e_1 e_2}(e_1).$$

2. The Fibonacci Polynomials

Note that if k is a real variable x, then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by [24]

$$F_{n+1}(x) = \begin{cases} 1 & \text{if } n = 0, \\ x & \text{if } n = 1, \\ xF_n(x) + F_{n-1}(x) & \text{if } n \geq 2. \end{cases} \tag{2.1}$$

from where the first Fibonacci polynomials are

$$\begin{aligned} F_1(x) &= 1, \\ F_2(x) &= x, \\ F_3(x) &= x^2 + 1, \\ F_4(x) &= x^3 + 2x, \\ F_5(x) &= x^4 + 3x^2 + 1, \\ F_6(x) &= x^5 + 4x^3 + 3x, \\ F_7(x) &= x^6 + 5x^4 + 6x^2 + 1, \\ F_8(x) &= x^7 + 6x^5 + 10x^3 + 4x. \end{aligned}$$

And from these expression, as for the k-Fibonacci numbers, we can write [24]:

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} x^{n-2i} \quad (n \geq 0).$$

Note that $F_{2n}(0) = 0$ and $x = 0$ is the only real root, while $F_{2n+1}(0) = 1$ with no real roots. Also for $x = k \in \mathbb{N}$, we obtain the elements of the k-Fibonacci numbers.

By iterating recurrence relation of formula (2.1), the following property is straightforwardly deduced.

Proposition 2.1 ([26]). *For* $1 \leq r \leq n-1$ *holds*:

$$F_{n+1}(x) = F_r(x)F_{n-(r-2)}(x) + F_{r-1}(x)F_{n-(r-1)}(x).$$

Proposition 2.2 (Binet's formula). *The nth Fibonacci polynomial may be written as*

$$F_n(x) = \frac{\sigma^n - (-\sigma)^{-n}}{\sigma + \sigma^{-1}}, \quad \textit{being } \sigma = \frac{x + \sqrt{x^2+4}}{2}. \tag{2.2}$$

Proof. Note that the characteristic equation for k-Fibonacci polynomials is $r^2 - x \cdot r - 1 = 0$ with roots $r_1 = \sigma = \frac{x+\sqrt{x^2+4}}{2}$, and $r_2 = -\sigma^{-1}$, from where Formula (2.2) is deduced. □

Proposition 2.3 ([25]). (*Asymptotic behavior of the quotient of consecutive terms*).

If $\sigma = \frac{x+\sqrt{x^2+4}}{2}$, then $\lim_{n\to\infty} \frac{F_{n+1}(x)}{F_n(x)} = \sigma$.

As a consequence, the quotient between two consecutive terms of the k-Fibonacci numbers $\{F_{k;n}\} = \{0, 1, k, k^2+1, k^3+2k, \ldots\}$ tends to the positive characteristic root $\sigma = \sigma_k$. For each integer k, $\sigma + \sigma_k$ is called the kth metallic ration [28]: Golden Ratio, for $k = 2$, and Bronze Ratio for $k = 3$.

Proposition 2.4 ([24]). (*Honsberger's formula*). *for* n, m *integers it holds*:

$$F_{m+n}(x) = F_{m-1}(x)F_n(x) + F_m(x)F_{n-1}(x).$$

3. On the Symmetric Functions of Some k-Fibonacci Numbers and Fibonacci Polynomials

In this part, we are now in a position to provide Theorem 3.1. Also, we derive the new generating functions of the products of some known numbers and polynomails.

Definition 3.1. The symmetrizing operator $\delta^{-k}_{e_1e_2}$ is defined by [7]

$$\delta^{-k}_{e_1e_2}(f\ ) = \frac{e_2^k f(e_1) - e_1^k f(e_2)}{(e_1e_2)^k(e_1 - e_2)} \quad (k \in \mathbb{N}).$$

Lemma 3.1 ([7]). *Let* $E = \{e_1, e_2\}$, *we define the operator* $\delta^{-k}_{e_1e_2}$ *as follows*:

$$\delta^{-k}_{e_1e_2} f\ (e_1) = \frac{-h_{k-1}(e_1,e_2)}{e_1^k e_2^k} f(e_1) + \frac{e_1^k}{e_1^k e_2^k}\partial_{e_1e_2} f(e_1).$$

Theorem 3.1. *Let E and A be two alphabets, respectively, $\{\epsilon_1, e_2\}$ and $\{a_1, a_2, \ldots\}$, then we have*

$$\frac{\sum_{n=0}^{\infty} e_n(a_1, a_2, \ldots, a_n)\delta_{e_1e_2}^{k+n-1}(e_1)z^n}{\prod_{a\in A}(1-ae_1z)\prod_{a\in A}(1-ae_2z)} = \sum_{n=0}^{k-1} h_n(a_1, a_2, \ldots, a_n)e_1^n e_2^n \delta_{e_1e_2}^{k-n-1}(e_1)z^n - e_1^k e_2^k z^{k+1}\sum_{n=0}^{\infty} h_{n+k+1}(a_1, a_2, \ldots, a_n)\delta_{e_1e_2}^{n}(e_1)z^n \tag{3.1}$$

for all $k \in \mathbb{N}$.

Proof. By applying the operator $\delta_{e_1e_2}^{-k}$ to the series $f(e_1) = (\prod_{a\in A}(1-ae_1z))^{-1}$, we have

$$\begin{aligned}
\delta_{e_1e_2}^{-k} f(e_1) &= \delta_{e_1e_2}^{-k}\left(\sum_{n=0}^{\infty} e_n(a_1, a_2, \ldots, a_n)e_1^n z^n\right)^{-1} \\
&= \frac{\frac{e_2^k}{\sum_{n=0}^{\infty} e_n(a_1,a_2,\ldots,a_n)e_1^n z^n} - \frac{e_1^k}{\sum_{n=0}^{\infty} e_n(a_1,a_2,\ldots,a_n)e_2^n z^n}}{e_1^k e_2^k (e_1-e_2)} \\
&= \frac{\sum_{n=0}^{\infty} e_n(a_1, a_2, \ldots, a_n)e_2^{n+k}z^n - \sum_{n=0}^{\infty} e_n(a_1, a_2, \ldots, a_n)e_1^{n+k}z^n}{e_1^k e_2^k (e_1-e_2)(\prod_{a\in A}(1-ae_1z))(\prod_{a\in A}(1-ae_2z))} \\
&= \frac{-1}{e_1^k e_2^k}\left(\frac{\sum_{n=0}^{\infty} e_n(a_1, a_2, \ldots, a_n)\delta_{e_1e_2}^{k+n-1}(e_1)z^n}{\prod_{a\in A}(1-ae_1z)\prod_{a\in A}(1-ae_2z)}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\delta_{e_1e_2}^{-k} f(e_1) &= \delta_{e_1e_2}^{-k}\left(\sum_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_n)e_1^n z^n\right) \\
&= \frac{e_2^k \sum_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_n)e_1^n z^n - e_1^k \sum_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_n)e_2^n z^n}{e_1^k e_2^k (e_1-e_2)} \\
&= \frac{1}{e_1^k e_2^k}\left(\sum_{n=0}^{\infty} h_n(a_1, a_2, \ldots, a_n)\frac{e_2^k e_1^n - e_1^k e_2^n}{e_1-e_2}z^n\right) \\
&= \frac{1}{e_1^k e_2^k}\left(\sum_{n=0}^{k-1} h_n(a_1, a_2, \ldots, a_n)\frac{e_2^k e_1^n - e_1^k e_2^n}{e_1-e_2}z^n\right. \\
&\quad \left. + \sum_{n=k+1}^{\infty} h_n(a_1, a_2, \ldots, a_n)\frac{e_2^k e_1^n - e_1^k e_2^n}{e_1-e_2}z^n\right)
\end{aligned}$$

$$= \frac{-1}{e_1^k e_2^k}\left(\sum_{n=0}^{k-1} h_n(a_1, a_2, \ldots, a_n) e_1^n e_2^n\, \delta_{e_1 e_2}^{k-n-1}(e_1) z^n - e_1^k e_2^k z^{k+1} \right.$$
$$\left. \times \sum_{n=0}^{\infty} h_{n+k+1}(a_1, a_2, \ldots, a_n) \delta_{e_1 e_2}^{n}(e_1) z^n \right).$$

This completes the proof. □

We now derive the new generating functions of the products of some known polynomials. Indeed, we consider Theorem 1 in order to derive k-Fibonacci numbers and Fibonacci polynomials if $k = 1$.

Theorem 3.2. *Let E and A be two alphabets, respectively, $\{e_1, e_2\}$ and $\{a_1, a_2\}$, then we have*

$$\sum_{n=0}^{\infty} h_{n+2}(a_1, a_2) \delta_{e_1 e_2}^{n}(e_1) z^n$$
$$= \frac{e_1 e_2 a_1^2 a_2^2 z^2 - a_1 a_2 h_1(e_1, e_2) h_1(a_1, a_2) z + (a_1 + a_2)^2 - a_1 a_2}{\prod_{a \in A}(1 - a e_1 z) \prod_{a \in A}(1 - a e_2 z)}. \tag{3.2}$$

In the case $A = \{a_1\}$, based on Theorem 3.2, we deduce the following lemmas:

Lemma 3.2. *Given two alphabets $E = \{e_1, -e_2\}$ and $A = \{a_1\}$, we have*

$$\sum_{n=0}^{\infty} a_1^n \delta_{e_1[-e_2]}^{n}(e_1) z^n = \frac{1}{1 - a_1(e_1 - e_2)z - a_1^2 e_1 e_2 z^2}. \tag{3.3}$$

Lemma 3.3. *Given two alphabets $E = \{e_1, -e_2\}$ and $A = \{a_1\}$, we have*

$$\sum_{n=0}^{\infty} a_1^n \delta_{e_1[-e_2]}^{n+1}(e_1) z^n = \frac{e_1 - e_2 + e_1 e_2 z}{1 - a_1(e_1 - e_2)z - a_1^2 e_1 e_2 z^2}. \tag{3.4}$$

Assuming that $e_1 - e_2 = 1$, $e_1 e_2 = 1$ and $a_1 = 1$ in Eqs. (3.3) and (3.4), we obtain the generating functions given by Boussayoud *et al.* [2, 5] which represent:

(1) The generating function of the Fibonacci numbers F_n.
(2) The generating function of the Lucas numbers L_n.

Choosing e_1 and e_2 such that $\left\{\begin{smallmatrix} e_1 e_2 = 1 \\ e_1 - e_2 = k \end{smallmatrix}\right.$ and substituting in (3.3) and (3.4) we end up with [3]

$$\sum_{n=0}^{\infty} \delta_{e_1[-e_2]}^{n}(e_1) z^n = \frac{1}{1 - kz - z^2} \tag{3.5}$$

and

$$\sum_{n=0}^{\infty} \delta^{n+1}_{e_1[-e_2]}(e_1)z^n = \frac{k+z}{1-kz-z^2}. \tag{3.6}$$

Thus, we deduce the following theorem.

Theorem 3.3 ([2]). *For* $n \in \mathbb{N}$, *the generating function of the* k*-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F_{k,n}z^n = \frac{1}{1-kz-z^2}.$$

Multiplying Eq. (3.5) by the variable $2+k^2$ and subtract it from (3.6) multiplying by the variable k yields

$$\sum_{n=0}^{\infty}[(2+k^2)\delta^{n}_{e_1[-e_2]}(e_1) - k\ \delta^{n+1}_{e_1[-e_2]}(e_1)]z^n = \frac{2-kz}{1-kz-z^2},$$

and we have the following theorem.

Theorem 3.4 ([2]). *For* $n \in \mathbb{N}$, *the generating function of the* k*-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} L_{k,n}z^n = \frac{2-kz}{1-kz-z^2}. \tag{3.7}$$

- Put $k=2$ in the relationship (3.7), we have

$$\sum_{n=0}^{\infty} Q_n t^n = \frac{2-2t}{1-2t-t^2},$$

which represents a generating function for Pell–Lucas numbers [5].

Choosing e_1 and e_2 such that $\begin{cases} e_1e_2 = 1 \\ e_1 - e_2 = x \end{cases}$ and substituting in (3.3), we end up with

$$\sum_{n=0}^{\infty} \delta^{n}_{e_1[-e_2]}(e_1)z^n = \frac{1}{1-xz-z^2}, \quad \text{with } e_1 = \sigma = \frac{x+\sqrt{x^2+4}}{2}.$$

Thus, we get the following theorem.

Theorem 3.5. *We have the following a new generating function of the Fibonacci polynomials as*

$$\sum_{n=0}^{\infty} F_n(x)t^n = \frac{1}{1-xt-t^2}.$$

For the case $E = \{e_1, -e_2\}$ and $A = \{a_1, -a_2\}$ with replacing e_2 by $-e_2$, a_2 by $-a_2$ in (3.2), we have

$$\sum_{n=0}^{\infty} h_{n+2}(a_1, [-a_2])h_n(e_1, [-e_2])z^n = \frac{-e_1e_2a_1^2a_2^2z^2 + a_1a_2h_1(e_1, [-e_2])h_1(a_1, [-a_2])z + (a_1 - a_2)^2 + a_1a_2}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}. \tag{3.8}$$

This case consists of three related parts. Firstly, the substitutions

$$\begin{cases} a_1 - a_2 = k, \\ a_1a_2 = 1, \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = x, \\ e_1e_2 = 1, \end{cases}$$

in (3.8) give

$$\sum_{n=0}^{\infty} h_{n+2}(a_1, [-a_2])h_n(e_1, [-e_2])z^n = \frac{k^2 + 1 + kxz - z^2}{1 - kxz - (x^2 + k^2 + 2)z^2 - kxz^3 + z^4} = \sum_{n=0}^{\infty} F_{k,n+2}F_n(x)z^n.$$

From which we have the following theorem.

Theorem 3.6. *We have the following a new generating function of the product of k-Fibonacci numbers and Fibonacci polynomials as*

$$\sum_{n=0}^{\infty} F_{k,n+2}F_n(x)z^n = \frac{k^2 + 1 + kxz - z^2}{1 - kxz - (x^2 + k^2 + 2)z^2 - kxz^3 + z^4}. \tag{3.9}$$

- Put $k = 1$ in the relationship (3.9), we have

$$\sum_{n=0}^{\infty} F_{n+2}F_n(x)z^n = \frac{2 + xz - z^2}{1 - xz - (x^2 + 3)z^2 - xz^3 + z^4},$$

  which represents a new generatings functions of the product of Fibonacci numbers and Fibonacci polynomials.
- Put $k = 2$ in the relationship (3.9), we have

$$\sum_{n=0}^{\infty} P_{n+2}F_n(x)z^n = \frac{5 + 2xz - z^2}{1 - 2xz - (x^2 + 6)z^2 - 2xz^3 + z^4},$$

  which represents a new generatings functions of the product of Pell numbers and Fibonacci polynomials.

Secondly, by making the following restrictions:

$$\begin{cases} a_1 - a_2 = k, \\ a_1 a_2 = 1, \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = k, \\ e_1 e_2 = 1, \end{cases}$$

in (3.8) gives

$$\begin{aligned} \sum_{n=0}^{\infty} h_{n+2}(a_1, [-a_2]) h_n(e_1, [-e_2]) z^n &= \frac{k^2 + 1 + k^2 z - z^2}{1 - k^2 z - 2(k^2+1) z^2 - k^2 z^3 + z^4} \\ &= \sum_{n=0}^{\infty} F_{k,n+2} F_{k,n} z^n \end{aligned} \tag{3.10}$$

representing a new generating function of k-Fibonacci numbers $F_{k,n}$.

On the other hand, we consider

$$\begin{aligned} \sum_{n=0}^{\infty} F_{k,n+2} F_{k,n} z^n &= \sum_{n=0}^{\infty} (F_{k,n+1} + F_{k,n}) F_{k,n} z^n \\ &= \sum_{n=0}^{\infty} F_{k,n+1} F_{k,n} z^n + \sum_{n=0}^{\infty} F_{k,n}^2 z^n. \end{aligned}$$

Since

$$\sum_{n=0}^{\infty} F_{k,n}^2 z^n = \frac{1 - z^2}{1 - k^2 z - 2(k^2+1) z^2 - k^2 z^3 + z^4} \quad \text{(see [3])}$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty} F_{k,n+2} F_{k,n} z^n &= \frac{k^2(1+z)}{1 - k^2 z - 2(k^2+1) z^2 - k^2 z^3 + z^4} \\ &\quad + \frac{1 - z^2}{1 - k^2 z - 2(k^2+1) z^2 - k^2 z^3 + z^4}, \end{aligned}$$

from those applications, we deduce the following theorem.

Theorem 3.7. *We have the following a new generating function of the product of two consecutive k-Fibonacci numbers as*

$$\sum_{n=0}^{\infty} F_{k,n+1} F_{k,n} z^n = \frac{k^2(1+z)}{1 - k^2 z - 2(k^2+1) z^2 - k^2 z^3 + z^4}. \tag{3.11}$$

- Put $k = 1$ in the relationship (3.10) and (3.11), we obtain the following results.

Corollary 3.1. *We have the following a new generating function of the product of Fibonacci numbers as*

$$\sum_{n=0}^{\infty} F_{n+2} F_n z^n = \frac{2 + z - z^2}{1 - z - 4z^2 - z^3 + z^4}.$$

Corollary 3.2. *For $n \in \mathbb{N}$, the generating function of the product of two consecutive Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} F_{n+1}F_n z^n = \frac{1}{1-2z-2z^2+z^3}.$$

- Put $k=2$ in the relationship (3.10) and (3.11), we obtain the following results.

Corollary 3.3. *We have the following a new generating function of the product of Pell numbers as*

$$\sum_{n=0}^{\infty} P_{n+2}P_n z^n = \frac{5+4z-z^2}{1-4z-10z^2-4z^3+z^4}.$$

Corollary 3.4. *We have the following a new generating function of the product of two consecutive Pell numbers as*

$$\sum_{n=0}^{\infty} P_{n+1}P_n z^n = \frac{4}{1-5z-5z^2+z^3}.$$

4. Conclusion

In this paper, we have derived new theorems in order to determine generating functions of k-Fibonacci numbers, k-Lucas numbers and Fibonacci polynomials. The derived theorems and lemmas are based on symmetric functions and products of these numbers and polynomials.

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