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# Characterization of Bi-Starlike Functions: A Daehee Polynomial Approach

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**Abstract:** This research investigates the second Hankel determinant for a specific class of functions associated with the Daehee polynomial. To achieve this, we introduce new subclasses of starlike functions in the context of Daehee polynomials. In complex analysis, establishing precise bounds for coefficient estimates in bi-univalent functions is essential, as these coefficients define the fundamental properties of conformal mappings. In this study, we derive sharp bounds for coefficient estimates within new subclasses of starlike functions related to Daehee polynomials, with most of the obtained limits demonstrating high accuracy. This work aims to inspire further exploration of rigorous bounds for analytic functions associated with innovative mapping domains.

**Keywords:** bi-univalent functions; best estimates; Daehee polynomial; Hankel determinant; starlike functions



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## 1. Introduction

In engineering, designing and optimizing complex systems such as optical and control systems pose significant challenges. To accurately model intricate wavefronts, engineers use specialized functions that fulfill precise optical criteria. In signal processing, univalent functions play a critical role in beamforming, enabling the manipulation of electromagnetic waves. Control systems engineering relies on univalent functions ( $\mathcal{UF}$ s) for filter design to achieve specific frequency responses while ensuring system stability and minimal phase distortion. Similarly, in mechanical systems, univalent functions help model system dynamics, identifying essential parameters for performance optimization. In robotics, these functions are also beneficial in managing manipulators and setting constraints on joint angles and velocities. Moreover, bi-univalent functions ( $\mathcal{BUs}$ ) are instrumental in improving compression ratios in image processing, enhancing image quality during compression and transmission, a persistent engineering challenge (see [1,2] for further details). The relationship between convex and starlike functions within Geometric Function Theory (GFT), initially identified by Alexander [3], has opened new avenues for investigation. Later, Robertson [4] expanded on this by presenting new approaches to understanding these classes of functions. A significant aspect of geometric theory in analytic functions ( $\mathcal{AF}$ s) is the coefficient problem, with extensive research focused on maximizing values of the 3rd Hankel determinant, as can be seen in [5–9].

To understand the main results, we begin by introducing some foundational concepts. Specifically, we consider  $\mathcal{AF}$  functions that meet particular conditions. These functions are defined in the open unit disk  $\mathcal{U}$ , which includes all complex numbers  $z$  with an absolute

value less than 1. The functions must satisfy the conditions  $\mu(0) = 0$  and  $\mu'(0) = 1$ . Additionally, they belong to the class  $\mathcal{A}$  and can be represented using a Taylor series:

$$\mu(z) = \sum_{\zeta=2}^{\infty} a_{\zeta} z^{\zeta} + z, \quad z \in \mathcal{U}. \quad (1)$$

Class  $\mathcal{S}$  comprises all functions that are one-to-one and satisfying specific normalization conditions:  $\mu(0)$  equals 0 and  $\mu'(0)$  equals 1. This class has served as the basis for ongoing research in this area.

Typically, polynomial functions are connected to a group of  $\mathcal{AF}$ s known as  $\mathcal{P}$ . This class contains normalized  $\mathcal{AF}$ s within the domain  $\mathcal{U}$  and takes a specific form as follows:

$$p(z) = \sum_{\zeta=2}^{\infty} r_{\zeta} z^{\zeta} + 1. \quad (2)$$

Regarding the function  $p(z)$ , we meet the condition  $\Re(p(z)) > 0$ , and  $p(0) = 1$ , both of which are fulfilled.

In the field of geometry, functions known as Carathéodory functions, or functions with positive real parts, play a crucial role and have strong links to almost all types of  $\mathcal{UF}$ s. From a geometric view, various subclasses of univalent functions have been established based on the shape of their image domains. In particular, the families  $\mathcal{S}^*$ ,  $\mathcal{K}$ , and  $\mathcal{BT}$  represent the class of starlike functions, the class of convex functions, and the class of bi-starlike functions, respectively, as defined by:

$$\mathcal{S}^* = \left\{ \mu : \mu \in \mathcal{S} \text{ and } \Re\left(\frac{z\mu'(z)}{\mu(z)}\right) > 0 \quad (z \in \mathcal{U}) \right\},$$

$$\mathcal{K} = \left\{ \mu : \mu \in \mathcal{S} \text{ and } \Re\left(1 + \frac{z\mu''(z)}{\mu'(z)}\right) > 0 \quad (z \in \mathcal{U}) \right\},$$

and

$$\mathcal{BT} = \{ \mu : \mu \in \mathcal{S} \text{ and } \Re(\mu'(z)) > 0 \quad (z \in \mathcal{U}) \}.$$

In the field of mathematics, examining Hankel determinants within specific classes provides a distinct viewpoint that reveals fundamental characteristics and behaviors of functions, which has important consequences for both theoretical and practical mathematics. For the particular inputs  $s_1$  and  $s_2$ , where  $s_1$  and  $s_2$  are natural numbers, Pommerenke [10,11] defined the Hankel determinant  $\mathcal{H}_{s_1, s_2}(f)$  for a function  $\mu \in \mathcal{S}$  that conforms to Equation (1) as follows:

$$\mathcal{H}_{s_1, s_2}(\mu) = \begin{vmatrix} a_{s_2} & a_{s_2+1} & a_{s_2+2} & \cdots & a_{s_2+s_1-1} \\ a_{s_2+1} & a_{s_2+2} & a_{s_2+3} & \cdots & a_{s_2+s_1} \\ a_{s_2+2} & a_{s_2+3} & a_{s_2+4} & \cdots & a_{s_2+s_1+1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{s_2+s_1-1} & a_{s_2+s_1} & a_{s_2+s_1+1} & \cdots & a_{s_2+2(s_1-1)} \end{vmatrix}. \quad (3)$$

Setting  $s_2 = 1$  and  $s_1 = 2$ ;

$$\mathcal{H}_{2,1}(\mu) = a_1 a_3 - a_2^2 = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}. \quad (4)$$

The determinant  $\mathcal{H}_{2,1}(\mu)$  is a type of Fekete-Szegő functional, as mentioned in [12]. Specifically, we are considering the case where  $s_2$  equals 2 and  $s_1$  equals 2:

$$\mathcal{H}_{2,2}(\mu) = a_2 a_4 - a_3^2 = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}. \quad (5)$$

For more details see [13–15].

In the realm of GFT,  $\mathcal{BF}s$  occupy a pivotal position due to their intricate relationship with  $\mathcal{UF}s$ , which are injective and analytic in a defined domain. This group of  $\mathcal{UF}s$  is known for its symmetrical properties that involve both the original function and its inverse, presents a more comprehensive perspective on conformal mapping and analytic properties.

One of the well-known issues in the field of GFT, which has brought crucial details on geometric requirements of functions, involves the bounds of the well-known coefficients  $|a_n|$ . The research on  $\mathcal{BF}s$ , denoted by  $\Sigma$ , was first investigated by Lewin [16], revealing that  $|a_2| < 151/100$  for functions allocated in  $\Sigma$ . After these successful research by Lewin [17], Brannan and Clunie [17] revealed that  $|a_2| \leq 1.414214$  and Netanyahu [18] demonstrated that for every  $\mu \in \Sigma$ , the maximum value of  $|a_2|$  is  $4/3$ . Recent research has focused on functions that are bi-univalent, particularly in getting non-sharp bounds for the coefficients  $|a_2|$  and  $|a_3|$ . Notably, Srivastava et al. [19] have made a groundbreaking contribution by revitalizing and significantly expanding the research on  $\mathcal{AF}s$  in few years back. Their work introduced bounds for  $|a_2|$  and  $|a_3|$ , which have been further explored by several researchers (see [20–24] and cited therein).

According to a fundamental theorem by Koebe, every function  $\mu$  in the family  $\mathcal{S}$  maps the unit disc onto regions that contains at least as a disc with a diameter of one-half. Consequently, all functions in  $\mathcal{S}$  possess an inverse, which can be defined on the same-sized disc. The expression for the inverse function of any given function  $\mu \in \mathcal{S}$  is:

$$\mu^{-1}(z) := g(\omega) = \omega - a_2\omega^2 - (a_3 - 2a_2^2)\omega^3 + (5a_2a_3 - 5a_2^3 - a_4)\omega^4 + \dots \quad (6)$$

Current research is primarily centered on studying functions from the same class that are associated with specific orthogonal polynomials.

In 2016, Güney [25] utilized the Chebyshev polynomial to establish a subset of  $\mathcal{BUs}$  and determine the absolute values of  $|a_2|$  and  $|a_3|$  for the subclass, as well as the Fekete-Szegő functional for the same subclass. Hamidi and Jahangiri [26] identify specific subclasses of  $\mathcal{BUs}$  with the help of Feber polynomials, while also demonstrating the erratic nature of the initial coefficients. To find newer research on  $\mathcal{BUs}$  connected to Feber polynomials, consult [27]. In 2018, Srivastava et al. [28] utilized Horadam polynomials to create a subclass of  $\mathcal{BUs}$  and achieved intriguing findings. To find newer research on  $\mathcal{BUs}$  connected to Horadam polynomials, consult [29]. Amourah et al. [30] introduce two novel subclasses of  $\mathcal{BUs}$  by employing Gegenbauer Polynomials. Following that, multiple researchers have investigated properties associated with the Gegenbauer Polynomials, such as [31]. In 2023, Shaba et al. [32] introduce a novel subclass of  $\mathcal{BUs}$  based on  $\tau$ -pseudo subordinate to Euler polynomial, yielding sharp results. In the same year, the introduction of novel subclasses of  $\mathcal{BUs}$  was achieved by Aktaş [33] through the utilization of generalized bivariate Fibonacci polynomials, resulting in the discovery of fresh outcomes.

Special polynomials and numbers have significant applications across various fields, including physics, engineering, mathematics, and related disciplines. These areas encompass topics such as mathematical analysis, mathematical physics, quantum mechanics, functional analysis, and differential equations, among others ([34–37]). These numbers and polynomials can be explored using a variety of approaches, including umbral calculus techniques, combinatorial methods, probability theory, differential equations, and generating functions. Among the notable families of polynomials and numbers is the Daehee family ([34–39]).

The Daehee polynomial is commonly defined, with a reference provided in [40], through the use of generating functions:

$$V(l, h) = \sum_{\zeta} R_{\zeta}(l) \frac{h^{\zeta}}{\zeta!} = \frac{\ln(1+h)}{h} (1+h)^l, \quad (7)$$

where  $h$  is complex or real. If  $1 + h$  becomes a complex number, the value of  $\ln(1 + h)$  is then defined as:

$$\ln(1 + h) = \ln|1 + h| + i \arg(1 + h),$$

where:

- $\ln|1 + h|$  is the real-valued logarithm of the modulus  $|1 + h|$ ,
- $\arg(1 + h)$  is the angle of  $1 + h$ , typically restricted to the principal branch  $(-\pi, \pi]$ .
- If  $h$  is real, restrict  $h > -1$ .

We recursively define the sequence of polynomials  $R_\zeta(l)$  as follows:

$$R_\zeta(l) = \sum_{\kappa=0}^{\zeta} \overline{\left[ \begin{matrix} \zeta \\ \kappa \end{matrix} \right]} B_\kappa(l) \quad (8)$$

where

$$\overline{\left[ \begin{matrix} \zeta \\ \kappa \end{matrix} \right]}$$

are Stirling numbers of the first kind, and  $B_\kappa(l)$  are Bernoulli numbers, respectively. The inverse relation of this equation is

$$B_\kappa(l) = \sum_{\zeta=0}^{\kappa} \left\{ \begin{matrix} \zeta \\ \kappa \end{matrix} \right\} R_\zeta(l). \quad (9)$$

where

$$\left\{ \begin{matrix} \zeta \\ \kappa \end{matrix} \right\}$$

are the Stirling numbers of the second kind.

A few Daehee polynomials are listed below:

$$\begin{aligned} R_0(l) &= 1, \\ R_1(l) &= l - \frac{1}{2}, \\ R_2(l) &= l^2 - 2l + \frac{2}{3}, \\ R_3(l) &= l^4 + 8l^3 + 21l^2 - 20l + \frac{24}{5}. \end{aligned} \quad (10)$$

In the past literature review under subclasses of analytic functions, there are no research work in this field of study on Daehee polynomials which really makes this work novel because with this Daehee polynomials on bi-univalent functions will open new directions of research in this field. This paper have been able to make use of the starlike functions with the Daehee polynomials in terms of the bi-univalent functions. This research did not make use of the existing polynomials in GFT, making the subclass a novel one in the field of GFT, which also opens up some new directions of research in this field. This research also successfully obtained the extremal function for this new class of bi-univalent functions, confirming first that this new class is not an empty class at all. Additionally, this extremal function validates each of the sharp bounds that have been obtained in this research successfully, making this research interesting and valuable to readers in the field of GFT. Also, due to the fact that no researchers have been able to develop this new subclass in the area of bi-univalent functions, it is impossible to compare these new results with other existing results. This research have successfully validated each of their results by comparing them with extremal functions, which are more efficient than the conventional way past researchers have done it-comparing the new results with existing results that may not be correct in some cases.

Recent advancements in the study of analytic functions have led to significant expansions through diverse methodologies. Among these, quantum calculus has emerged

as a prominent tool due to its extensive applications across various scientific disciplines. Building on prior research, this study introduces a novel class of bi-starlike functions subordinate to the Daehee polynomial. A key focus is placed on determining the coefficient values ( $|a_n|$ ) for  $n = 2, 3, 4$ , which are further utilized to analyze second-order Hankel determinants and the Fekete–Szegő functional. This comprehensive investigation provides deeper insights into these mathematical constructs.

To provide a foundation for this study, we first introduce a key definition that characterizes the class of functions central to our analysis.

**Definition 1.** If the expression  $\mu$ , as defined in (6), is a member of the class  $SD_{\Sigma}(l)$ , then

$$\frac{z\mu'(z)}{\mu(z)} \prec V(l, z)$$

and

$$\frac{\omega[\mu^{-1}(z)]'}{\mu^{-1}(z)} \prec V(l, \omega)$$

are satisfied.

The lemmas mentioned below are crucial to be addressed in this paper because they have significant implications in the derivations of each theorem within the main outcomes.

**Lemma 1** ([41]). Specifically, consider  $p \in P$ , which represents a series of some form

$$p(z) = 1 + \sum_{n=1}^{\infty} r_{\zeta} z^{\zeta} \quad (11)$$

where  $\Re(p(z)) > 0$  and  $z$  in  $\mathcal{U}$ , then

$$|r_{\zeta}| \leq 2, \quad \zeta \in \mathbb{N}.$$

**Lemma 2** ([41]). Given that  $p \in P$  can be expressed as a series according to Equation (11), and  $\Re(p(z)) > 0$  holds true for all  $z$  in  $\mathcal{U}$ , then

$$r_2 = \frac{r_1^2 + (4 - r_1^2)g_1}{2},$$

$$r_3 = \frac{r_1^3 + 2(4 - r_1^2)r_1g_1 - (4 - r_1^2)r_1g_1^2 + 2(4 - r_1^2)(1 - |g_1|^2)z}{4}.$$

## 2. Coefficients Bounds Results

In this part, we now present the estimates for the coefficients of  $|a_2|$ ,  $|a_3|$ , and  $|a_4|$  as provided by the following Theorem.

**Theorem 1.** Suppose  $\mu \in SD_{\Sigma}(l)$ . Then,

$$|a_2| \leq \frac{2l - 1}{2},$$

$$|a_3| \leq \frac{2l - 1}{4},$$

$$|a_4| \leq \frac{2l - 1}{6}.$$

The equations below prove that the coefficients bounds  $|a_2|$ ,  $|a_3|$ , and  $|a_4|$  are at their best estimates, which are the extremal function for the new class being studied.

$$\begin{aligned} \mu_1(z) &= z + \frac{2l-1}{2}z^2 + \dots, \\ \mu_2(z) &= z + \frac{2l-1}{4}z^3 + \dots, \\ \mu_3(z) &= z + \frac{2l-1}{6}z^4 + \dots. \end{aligned}$$

**Proof.** Suppose  $\mu \in \mathcal{SD}_\Sigma(l)$ . Then the following  $\mathcal{AF}$ s defined by  $E : \mathcal{U} \rightarrow \mathcal{U}$  alongside  $E_1(0) = 0 = E_2(0)$ ,  $|E(z)| < 1$  and  $|E(\omega)| < 1$ , fulfills the conditions below

$$\frac{z\mu'(z)}{\mu(z)} = V(l, E_1(z)) \tag{12}$$

and

$$\frac{\omega[\mu^{-1}(z)]'}{\mu^{-1}(z)} = V(l, E_2(\omega)). \tag{13}$$

Now, by utilizing the Caratheodory functions in conjunction with (12) and (13), and after some calculations, we obtain:

$$\begin{aligned} \frac{z\mu'(z)}{\mu(z)} &= 1 + \frac{R_1(l)}{2}r_1z + \left( \frac{R_1(l)}{2} \left( r_2 - \frac{r_1^2}{2} \right) + \frac{R_2(l)}{8}r_1^2 \right) z^2 \\ &+ \left( \frac{R_1(l)}{2} \left( r_3 - r_1r_2 + \frac{r_1^3}{4} \right) + \frac{R_2(l)}{4}r_1 \left( r_2 - \frac{r_1^2}{2} \right) + \frac{R_3(l)}{48}r_1^3 \right) z^3 + \dots \end{aligned} \tag{14}$$

and

$$\begin{aligned} \frac{\omega[\mu^{-1}(z)]'}{\mu^{-1}(z)} &= 1 + \frac{R_1(l)}{2}k_1\omega + \left( \frac{R_1(l)}{2} \left( k_2 - \frac{k_1^2}{2} \right) + \frac{R_2(l)}{8}k_1^2 \right) \omega^2 \\ &+ \left( \frac{R_1(l)}{2} \left( k_3 - k_1k_2 + \frac{k_1^3}{4} \right) + \frac{R_2(l)}{4}k_1 \left( k_2 - \frac{k_1^2}{2} \right) + \frac{R_3(l)}{48}k_1^3 \right) \omega^3 + \dots, \end{aligned} \tag{15}$$

where  $r, k \in P$ .

If the simplifications and rearrangements on the LHS of (14) and (15) are performed, the resulting equations for  $a_2$ ,  $a_3$ , and  $a_4$  will equalize the coefficients of terms of the same degree, to have:

$$a_2 = \frac{R_1(l)}{2}r_1, \tag{16}$$

$$2a_3 - a_2^2 = \frac{R_1(l)}{2} \left( r_2 - \frac{r_1^2}{2} \right) + R_2(l) \frac{r_1^2}{8}, \tag{17}$$

$$3a_4 - 3a_2a_3 + a_2^3 = \frac{R_1(l)}{2} \left( r_3 - r_1r_2 + \frac{r_1^3}{4} \right) + \frac{R_2(l)}{4}r_1 \left( r_2 - \frac{r_1^2}{2} \right) + \frac{R_3(l)}{48}r_1^3 \tag{18}$$

also,

$$-a_2 = \frac{R_1(l)}{2}k_1, \tag{19}$$

$$3a_2^2 - 2a_3 = \frac{R_1(l)}{2} \left( k_2 - \frac{k_1^2}{2} \right) + \frac{R_2(l)}{8}k_1^2, \tag{20}$$

$$-10a_2^3 + 12a_2a_3 - 3a_4 = \frac{R_1(l)}{2} \left( k_3 - k_1k_2 + \frac{k_1^3}{4} \right) + \frac{R_2(l)}{4} k_1 \left( k_2 - \frac{k_1^2}{2} \right) + \frac{R_3(l)}{48} k_1^3. \quad (21)$$

Now, in order for us to obtain the best estimate for the bound  $a_2$ , we will use (16) and (19) to derive the following expression:

$$\frac{R_1(l)}{2} r_1 = -\frac{R_1(l)}{2} k_1 \Rightarrow r_1 = -k_1 \Rightarrow r_1^2 = k_1^2 \Rightarrow r_1^3 = -k_1^3. \quad (22)$$

The best estimate for the bound  $a_2$  will be complete after applying Lemma 1 in (22), which makes the first outcome of the theorem very clear.

Next is to make  $a_3$  the subject of the formula by utilizing (17) and (20) in conjunction with considering  $r_1 = -k_1$ , we obtain

$$a_3 = a_2^2 + \frac{R_1(l)(r_2 - k_2)}{8};$$

then,

$$a_3 = \frac{R_1^2(l)r_1^2}{4} + \frac{R_1(l)(r_2 - k_2)}{8}. \quad (23)$$

Also, to get  $a_4$ , the subject of the formula, we make use of (18) and (21), alongside (22) and (23), and obtain:

$$a_4 = \frac{12R_1^3(l) + 6R_1(l) - 6R_2(l) + R_3(l)}{144} r_1^3 + \frac{15R_1^2(l)r_1(r_2 - k_2)}{96} + \frac{R_1(l)(r_3 - k_3)}{12} - \frac{(2R_1(l) - R_2(l))r_1(r_2 + k_2)}{24}. \quad (24)$$

In order to get the best estimate for  $a_3$  and  $a_4$ , we have the following expression by making use of (22) in Lemma 1 as follows:

$$r_2 - k_2 = \frac{4 - r_1^2}{2} (r_4 - k_4), \quad (25)$$

$$r_2 + k_2 = r_1^2 + \frac{4 - r_1^2}{2} (r_4 + k_4), \quad (26)$$

$$r_3 - k_3 = \frac{r_1^3}{2} + \frac{(4 - r_1^2)r_1}{2} (r_4 + k_4) - \frac{(4 - r_1^2)r_1}{4} (r_4^2 + k_4^2) + \frac{4 - r_1^2}{2} \left( [1 - |r_4|^2]z - [1 - |k_4|^2]w \right), \quad (27)$$

where  $|r_4| \leq 1$ ,  $|k_4| \leq 1$ ,  $|z| \leq 1$ , and  $|w| \leq 1$ .

Now, from (23), we substitute (25) to have:

$$a_3 = \frac{R_1^2(l)r_1^2}{4} + \frac{R_1(l)(4 - r_1^2)(r_4 + k_4)}{8}. \quad (28)$$

Note that if we define  $|r_1|$  as  $m$ , i.e., we make no loss of generality by assuming  $m$  is within the range  $[0, 2]$ . In this case, by applying the triangle inequality and setting  $|r_4|$  as  $r_5$  and  $|k_4|$  as  $k_5$ , we can formulate an inequality for  $|a_3|$  to be:

$$|a_3| \leq \frac{R_1^2(l)}{4} m^2 + \frac{R_1(l)(4 - m^2)(r_5 + k_5)}{16}, \quad (r_5, k_5) \in [0, 1]^2.$$

Define a function  $B : \mathbb{R}^2 \rightarrow \mathbb{R}$  as specified below:

$$B(r_5, k_5) = \frac{R_1^2(l)}{4} m^2 + \frac{R_1(l)(4 - m^2)(r_5 + k_5)}{16}, \quad (r_5, k_5) \in [0, 1]^2.$$

Now, let denote  $\Omega = \{(r_5, k_5) : (r_5, k_5) \in [0, 1]^2\}$  as a closed square, on which it is necessary for us to maximize the function  $B$ .

It is evident that  $B$ , which maps  $\mathbb{R}^2$  to  $\mathbb{R}$ , attains its greatest value at the edges or boundary of the closed square  $\Omega$ . The next step is to take the differentiation of  $B(r_5, k_5)$  with respect to  $r_5$  to have

$$B_{r_5}(r_5, k_5) = \frac{R_1(l)(4 - m^2)}{16}.$$

Since  $B_{r_5}(r_5, k_5) \geq 0$  is always non-negative ( $B_{r_5}(r_5, k_5) \geq 0$ ), it means that  $B(r_5, k_5)$  is an increasing function with respect to  $r_5$ , and its greatest value is reached when  $r_5 = 1$ , then

$$\max\{B(r_5, k_5) : r_5 \in [0, 1]\} = B(1, k_5) = \frac{R_1^2(l)}{4} m^2 + \frac{R_1(l)(4 - m^2)(1 + k_5)}{16},$$

where  $k_5 \in [0, 1]$  and  $m \in [0, 2]$ .

Now, we need to find the derivative of  $B(1, k_5)$ , which is given as follows:

$$B'(1, k_5) = \frac{R_1(l)(4 - m^2)}{16}.$$

Since  $B'(1, k_5) \geq 0$  is always non-negative ( $B'(1, k_5) \geq 0$ ), it means that  $B(1, k_5)$  is an increasing function and its greatest value is reached when  $k_5 = 1$ , then

$$\max\{B(1, k_5) : k_5 \in [0, 1]\} = B(1, 1) = \frac{R_1^2(l)}{4} m^2 + \frac{R_1(l)(4 - m^2)}{8}, \quad m \in [0, 2].$$

Hence, we obtain

$$\begin{aligned} B(r_5, k_5) &\leq \max\{B(r_5, k_5) : (r_5, k_5) \in \Omega\} = B(1, 1) \\ &= \frac{R_1^2(l)}{4} m^2 + \frac{R_1(l)(4 - m^2)}{8}. \end{aligned}$$

We can clearly see that  $|a_3| \leq B(r_5, k_5)$ , so we have:

$$|a_3| \leq c(l)m^2 + \frac{R_1(l)}{2}, \quad m \in [0, 2]$$

where

$$c(l) = \frac{1}{4} \left\{ R_1^2(l) - \frac{R_1(l)}{2} \right\}.$$

Let locate the greatest of the function  $W : \mathbb{R} \rightarrow \mathbb{R}$ , which is defined as follows

$$W(m) = c(l)m^2 + \frac{R_1(l)}{2}$$

in  $[0, 2]$ .

Now, by taking the derivative of  $W(m)$ , we obtain  $W'(m) = 2c(l)m$  where  $m \in [0, 2]$ . It is clear that  $W'(m) \leq 0$  whenever  $c(l) \leq 0$ , which implies that  $W'(m)$  is a decreasing function and takes its greatest at  $m = 0$ , then

$$\max\{W(m) : m \in [0, 2]\} = W(0) = \frac{R_1(l)}{2}$$

also  $W'(m) \geq 0$  whenever  $c(l, h, q) \geq 0$ , which implies that  $W'(m)$  is an increasing function and takes its greatest at  $m = 2$ , then

$$\max\{W(m) : m \in [0, 2]\} = W(2) = R_1^2(l).$$

Hence, we can now write the best estimate for  $|a_3|$  as:

$$|a_3| \leq \max\left\{R_1^2(l), \frac{R_1(l)}{2}\right\}.$$

Next, we need to obtain the best estimate for  $|a_4|$ , which can be determined by utilizing (24)–(27). Additionally, by applying the triangle inequality, we have:

$$|a_4| \leq v_1(m) + v_2(m)(r_5 + k_5) + v_3(m)(r_5^2 + k_5^2) = D(r_5, k_5), \quad (29)$$

where

$$\begin{aligned} v_1(m) &= \frac{12R_1^3(l) - 12R_2(l) + R_3(l)}{144}m^3 + \frac{R_1(l)(4 - m^2)}{12}, \\ v_2(m) &= \frac{15R_1(l)(4 - m^2)}{192}m + \frac{R_2(l)(4 - m^2)}{48}m, \\ v_3(m) &= \frac{R_1(l)(4 - m^2)(m - 2)}{48}. \end{aligned}$$

Now, let denote  $\Omega = \{(r_5, k_5) : (r_5, k_5) \in [0, 1]^2\}$  as a closed square, on which it is necessary for us to maximize the function  $D$  for each  $m \in [0, 2]$ .

It is so clear to see that the following coefficients  $v_1(m)$ ,  $v_2(m)$ , and  $v_3(m)$  of  $D(r_5, k_5)$  all depend on  $m$ , which prompts us to analyze the greatest of  $D$  at numerous values of  $m$ . Now, for the value  $m = 0$ , we have:

$$\begin{aligned} v_2(0) &= 0, \\ v_1(0) &= \frac{R_1(l)}{3} \quad \text{and} \\ v_3(0) &= -\frac{R_1(l)}{6}. \end{aligned}$$

We now substitute  $v_1(0)$  and  $v_3(0)$  into (29), to have:

$$D(r_5, k_5) = \frac{R_1(l)}{3} - \frac{R_1(l)}{6}(r_5^2 + k_5^2), \quad (r_5, k_5) \in [0, 1]^2.$$

Hence, we have

$$D(r_5, k_5) \leq \max\{D(r_5, k_5) : (r_5, k_5) \in \Omega\} = D(0, 0) = \frac{R_1(l)}{3}.$$

Also, for the value  $m = 2$ , we have:

$$\begin{aligned} v_2(2) &= v_3(2) = 0 \quad \text{and} \\ v_1(2) &= \frac{12R_1^3(l) - 12R_2(l) + R_3(l)}{18}. \end{aligned}$$

We now substitute  $v_1(2)$  into (29), to have:

$$D(r_5, k_5) = \frac{12R_1^3(l) - 12R_2(l) + R_3(l)}{18}.$$

Hence, we can now write the best estimate for  $|a_4|$  as:

$$|a_4| \leq \max \left\{ \frac{12R_1^3(l) - 12R_2(l) + R_3(l)}{18}, \frac{R_1(l)}{3} \right\}.$$

□

### 3. Estimations for the Second Hankel Determinant

The second Hankel determinant provides crucial insights into the structural properties of analytic and univalent functions, offering a means to evaluate their growth and distortion characteristics. So we give the following Theorem.

**Theorem 2.** Suppose  $\mu \in \mathcal{SD}_\Sigma(l)$ . Then,

$$|a_2a_4 - a_3^2| \leq \left( \frac{2l-1}{4} \right)^2.$$

The equation below prove that the coefficients bounds  $|a_2a_4 - a_3^2|$  is at its best estimates.

$$\mu_2(z) = z + \frac{2l-1}{4}z^3 + \dots$$

**Proof.** Suppose that  $\mu \in \mathcal{SD}_\Sigma(l)$ . We make use of (24), (25), (26), and (27) to obtain  $a_2a_4 - a_3^2$  to be:

$$\begin{aligned} a_2a_4 - a_3^2 = & \frac{-R_1^4(l) + 6R_1^2(l) - 6R_1(l)R_2(l) + R_1(l)R_3(l)}{288}r_1^4 + \frac{R_1^3(l)(r_2 - k_2)}{64}r_1^2 \\ & + \frac{R_1^2(l)(r_3 - k_3)}{24}r_1 - \frac{R_1(l)(2R_1(l) - R_2(l))(r_2 + k_2)}{48}r_1^2 - \frac{R_1^2(l)}{64}(r_5 + k_5)^2. \end{aligned}$$

Now, by setting  $|r_1| = m$ ,  $|r_4| = r_5$ , and  $|k_4| = k_5$ , and making use of (25), (26), and (27), we have the following expression for  $|a_2a_4 - a_3^2|$  to be:

$$|a_2a_4 - a_3^2| \leq V_1(m) + V_2(m)(r_5 + k_5) + V_3(m)(r_5^2 + k_5^2) + V_4(m)(r_5 + k_5)^2,$$

where

$$\begin{aligned} V_1(m) &= \frac{R_1^4(l) - R_1(l)R_3(l)}{288}m^4 + \frac{R_1^2(l)(4 - m^2)}{24}m \geq 0, \\ V_2(m) &= \frac{R_1^3(l)(4 - m^2)}{128}m^2 + \frac{R_1(l)R_2(l)(4 - m^2)}{96}m^2 \geq 0, \\ V_3(m) &= \frac{R_1^2(l)(4 - m^2)(m - 2)m}{96} \leq 0, \\ V_4(m) &= \frac{R_1^2(l)(4 - m^2)^2}{256} \geq 0. \end{aligned}$$

Define a function  $S : \mathbb{R}^2 \rightarrow \mathbb{R}$  as specified below:

$$S(r_5, k_5) = V_1(m) + V_2(m)(r_5 + k_5) + V_3(m)(r_5^2 + k_5^2) + V_4(m)(r_5 + k_5)^2, \quad (r_5, k_5) \in [0, 1]^2 \quad (30)$$

for each  $m \in [0, 2]$ .

Now, let denote  $\Omega = \{(r_5, k_5) : (r_5, k_5) \in [0, 1]^2\}$  as a closed square, on which it is necessary for us to greatest the function  $S$  for each  $m \in [0, 2]$ .

It is so clear to see that the following coefficients  $V_1(m)$ ,  $V_2(m)$ ,  $V_3(m)$  and  $V_4(m)$  of  $S(r_5, k_5)$  all depend on  $m$ , which prompts us to analyze the greatest of  $S$  at numerous values of  $m$ .

1. Now, for the value  $m = 0$ , we have:

$$V_1(0) = V_2(0) = V_3(0) = 0 \text{ and} \\ V_4(0) = \frac{R_1^2(l)}{16}.$$

We now substitute  $V_4(0)$  into (30), to have:

$$S(r_5, k_5) = \frac{R_1^2(l)}{16}(r_5 + k_5)^2, \quad (r_5, k_5) \in \Omega. \quad (31)$$

According to the equation in (31), it is immediately apparent that the function reaches its greatest at the edge of the region  $\Omega$ . The next step is to take the differentiation of  $S(r_5, k_5)$  with respect to  $r_5$  to have

$$S_{r_5}(r_5, k_5) = \frac{R_1^2(l)(r_5 + k_5)}{8}, \quad k_5 \in [0, 1].$$

Since  $S_{r_5}(r_5, k_5) \geq 0$  is always non-negative ( $S_{r_5}(r_5, k_5) \geq 0$ ), it means that  $S(r_5, k_5)$  is an increasing function with respect to  $r_5$ , and its greatest value is reached when  $r_5 = 1$ , then

$$\max\{S(r_5, k_5) : r_5 \in [0, 1]\} = S(1, k_5) = \frac{R_1^2(l)}{16}(1 + k_5)^2, \quad k_5 \in [0, 1].$$

Now, we need to find the derivative of  $S(1, k_5)$ , which is given as follows:

$$S'(1, k_5) = \frac{R_1^2(l)(r_5 + k_5)}{8}, \quad k_5 \in [0, 1].$$

Since  $S'(1, k_5) \geq 0$  is always non-negative ( $S'(1, k_5) \geq 0$ ), it means that  $S(1, k_5)$  is an increasing function and its greatest value is reached when  $k_5 = 1$ , then

$$\max\{S(1, k_5) : k_5 \in [0, 1]\} = S(1, 1) = \frac{R_1^2(l)}{4}.$$

Furthermore, for the particular parameter  $m = 0$ , we obtain:

$$S(r_5, k_5) \leq \max\{S(r_5, k_5) : (r_5, k_5) \in [0, 1]^2\} = S(1, 1) = \frac{R_1^2(l)}{4}.$$

Hence, for  $|a_2a_3 - a_3^2| \leq S(r_5, k_5)$ , we obtain

$$|a_2a_3 - a_3^2| \leq \frac{R_1^2(l)}{4},$$

for the case  $m = 0$ .

2. Now, for the value  $m = 2$ , we have:

$$V_2(2) = V_3(2) = V_4(2) = 0 \text{ and} \\ V_1(2) = \frac{R_1^4(l) - R_1(l)R_3(l)}{18}.$$

We now substitute  $V_1(2)$  into (30), to have:

$$S(r_5, k_5) = V_1(2) = \frac{R_1^4(l) - R_1(l)R_3(l)}{18}.$$

Thus, we obtain

$$|a_2a_3 - a_3^2| \leq \frac{R_1^4(l) - R_1(l)R_3(l)}{18},$$

for the case  $m = 2$ .

3. For this case, we take  $m \in (0, 2)$  and then analyze the greatest of  $S$  by taking note of these signs:  $\Delta(S) = S_{r_5r_5}(r_5, k_5)S_{k_5k_5}(r_5, k_5) - (S_{r_5k_5}(r_5, k_5))^2$ . Clearly, we can note that  $\Delta(S) = 4V_3(m)[V_3(m) + 2V_4(m)]$ , in which the sign  $\Delta(S)$  will be analyzed in two different instances.
  - 3.1. Suppose  $V_3(m) + 2V_4(m) \leq 0$  for the condition  $m \in (0, 2)$ . For this particular instance, since  $S_{r_5k_5}(r_5, k_5) = S_{k_5r_5}(r_5, k_5) = 2V_4(m) \geq 0$  and  $\Delta(S) \geq 0$ , then we can say that the function  $S$  can only have a minimum and not a maximum by applying elementary calculus on  $\Omega$ .
  - 3.2. Suppose  $V_3(m) + 2V_4(m) \geq 0$  for the condition  $m \in (0, 2)$ . For this particular instance, since  $\Delta(S) \leq 0$ , then we can say that the function  $S$  cannot take greatest on  $\Omega$ .

Thus, with the three instances investigated, we can write the best estimate for  $|a_2a_3 - a_3^2|$  to be

$$|a_2a_3 - a_3^2| \leq \max \left\{ \frac{R_1^2(l)}{4}, \frac{R_1^4(l) - R_1(l)R_3(l)}{18} \right\}.$$

□

#### 4. Estimations for the Fekete-Szegő Inequality

The Fekete-Szegő inequality is important because it gives an upper bound for the analytic functions by making the coefficients in the paper. So we state the following Theorem.

**Theorem 3.** Suppose  $\mu \in SD_{\Sigma}(l)$  and  $\gamma \in \mathbb{C}$ . Then,

$$|a_3 - \gamma a_2^2| \leq \begin{cases} Z(l), & |1 - \gamma| \leq Z(l) \\ \frac{(2l-1)|1-\gamma|}{2}, & |1 - \gamma| \geq Z(l). \end{cases}$$

where

$$Z(l) = \frac{2l-1}{4}. \quad (32)$$

The equation below prove that the Fekete-Szegő inequality  $|a_3 - \gamma a_2^2|$  is at is best estimates.

$$\mu_2(z) = z + \frac{2l-1}{4}z^3 + \dots$$

**Proof.** Suppose that  $\mu \in SD_{\Sigma}(l)$ . We make use of (22), (23) and, (25) to obtain  $|a_3 - \gamma a_2^2|$  to be:

$$a_3 - \gamma a_2^2 = (1 - \gamma) \frac{R_1^2(l)r_1^2}{4} + \frac{R_1(l)(4 - r_1^2)(r_4 - k_4)}{16} \quad (33)$$

for some  $r_4, k_4$  alongside  $|r_4| \leq 1$  and  $|k_4| \leq 1$ .

Now, by setting  $|r_1| = m$ ,  $|r_4| = r_5$ , and  $|k_4| = k_5$ , and applying triangle inequality on (33), we have the following expression for  $|a_3 - \gamma a_2^2|$  to be:

$$|a_3 - \gamma a_2^2| \leq \frac{|1 - \gamma|R_1(l)}{4}m^2 + \frac{R_1(l)(4 - m^2)(r_5 + k_5)}{16}, \quad (r_5, k_5) \in \Omega$$

for each  $m \in [0, 2]$ .

Define a function  $H : \mathbb{R}^2 \rightarrow \mathbb{R}$  as specified below:

$$H(r_5, k_5) = \frac{|1 - \gamma|R_1(l)}{4}m^2 + \frac{R_1(l)(4 - m^2)(r_5 + k_5)}{16}, \quad (r_5, k_5) \in \Omega$$

for each  $m \in [0, 2]$ .

Now, let denote  $\Omega = \{(r_5, k_5) : (r_5, k_5) \in [0, 1]^2\}$  as a closed square, on which it is necessary for us to maximize the function  $H$  for each  $m \in [0, 2]$ .

It is evident that  $H$  attains its greatest value at the edges or boundary of the closed square  $\Omega$ . The next step is to take the differentiation of  $H(r_5, k_5)$  with respect to  $r_5$  to have

$$H_{r_5}(r_5, k_5) = \frac{R_1(l)(4 - m^2)}{16} \quad m \in [0, 2].$$

Since  $H_{r_5}(r_5, k_5) \geq 0$  is always non-negative ( $H_{r_5}(r_5, k_5) \geq 0$ ), it means that  $H(r_5, k_5)$  is an increasing function with respect to  $r_5$ , and its greatest value is reached when  $r_5 = 1$ . Therefore,

$$\begin{aligned} \max\{H(r_5, k_5) : r_5 \in [0, 1]\} &= H(1, k_5) = \frac{|1 - \gamma|R_1(l)}{4}m^2 \\ &+ \frac{R_1(l)(4 - m^2)(1 + k_5)}{16}, \quad k_5 \in [0, 1], \end{aligned}$$

for each  $m \in [0, 2]$ .

Now, we need to find the derivative of  $H(1, k_5)$ , which is given as follows:

$$H'(1, k_5) = \frac{R_1(l)(4 - m^2)}{16} \quad m \in [0, 2].$$

Since  $H'(1, k_5) \geq 0$  is always non-negative ( $H'(1, k_5) \geq 0$ ), it means that  $H(1, k_5)$  is an increasing function and its greatest value is reached when  $k_5 = 1$ , then

$$\begin{aligned} \max\{H(1, k_5) : k_5 \in [0, 1]\} &= H(1, 1) = \frac{|1 - \gamma|R_1(l)}{4}m^2 \\ &+ \frac{R_1(l)(4 - m^2)}{8}, \quad m \in [0, 2]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} H(r_5, k_5) &\leq \max\{H(1, k_5) : k_5 \in [0, 1]\} = H(1, 1) \\ &= \frac{|1 - \gamma|R_1(l)}{4}m^2 + \frac{R_1(l)(4 - m^2)}{8}. \end{aligned}$$

Hence  $|a_3 - \gamma a_2^2| \leq H(r_5, k_5)$ , we have

$$|a_3 - \gamma a_2^2| \leq \frac{R_1(l)}{4} \left\{ |1 - \gamma| - Z(l) \right\} m^2 + Z(l),$$

where

$$Z(l) = \frac{R_1(l)}{2}.$$

Now we take the instance where we find the greatest of the function  $u : [0, 2] \rightarrow \mathbb{R}$  denoted as:

$$u(m) = \frac{R_1(l)}{4} \left\{ |1 - \gamma| - Z(l) \right\} m^2 + Z(l).$$

We can now take the derivative of  $u(m)$  to have:

$$u'(m) = \frac{R_1(l)}{2} \left\{ |1 - \gamma| - Z(l) \right\} m.$$

If  $u'(m) \leq 0$ , the function  $u(m)$  is decreasing. Suppose  $|1 - \gamma| \leq Z(l)$  and the greatest occurs at  $m = 0$ , then

$$\max\{u(m) : m \in [0, 2]\} = u(0) = Z(l).$$

and If  $u'(m) \geq 0$ , the function  $u(m)$  is increasing. Suppose  $|1 - \gamma| \geq Z(l)$  and the greatest occurs at  $m = 2$ , then

$$\max\{u(m) : m \in [0, 2]\} = u(2) = R_1(l)|1 - \gamma|.$$

Hence, we can now write the best estimate for  $|a_3 - \gamma a_2^2|$  as:

$$|a_3 - \gamma a_2^2| \leq \begin{cases} Z(l), & |1 - \gamma| \leq Z(l) \\ \frac{(2l-1)|1-\gamma|}{2}, & |1 - \gamma| \geq Z(l). \end{cases}$$

□

**Theorem 4.** Suppose  $\mu \in \mathcal{SD}_\Sigma(l)$  and  $\gamma \in \mathbb{R}$ . Then,

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{(2l-1)(1-\gamma)}{2} & \text{if } \gamma \leq 1 - Z(l) \\ Z(l) & \text{if } 1 - Z(l) \leq \gamma \leq 1 + Z(l) \\ \frac{(2l-1)(\gamma-1)}{2} & \text{if } 1 + Z(l) \leq \gamma. \end{cases} \quad (34)$$

where

$$Z(l) = \frac{2l-1}{4}.$$

**Proof.** Assuming  $\mu \in \mathcal{SD}_\Sigma(l)$ . We have  $|1 - \gamma| \geq Z(l)$  and  $|1 - \gamma| \leq Z(l)$  when  $\gamma \in \mathbb{R}$ . Yields:

$$\gamma \leq 1 - Z(l) \text{ either } \gamma \geq 1 + Z(l)$$

and

$$1 - Z(l) \leq \gamma \leq 1 + Z(l).$$

□

## 5. Conclusions

Numerous researchers have endeavored to classify  $\mathcal{BLUs}$  into specific subclasses. Our investigation has concentrated on starlike functions, denoted by  $\mathcal{SD}_\Sigma(l)$ , connected to Daehee polynomial. We obtained the best estimates for the coefficient problems. All our estimates were rigorously proven to be exact. Our key findings include the Fekete-Szegő

functional and the second Hankel determinants. Notably, our analysis yielded substantial improvements above more basic forms of these popular operations, showing a notable improvement in their efficiency. Additionally, investigating the geometric properties and applications of  $\mathcal{SD}_\Sigma(l)$  in mathematical physics and engineering could yield further insights. Exploring connections with modern techniques, such as fractional calculus and  $q$ -calculus, may also provide deeper understanding and novel applications. We believe our findings establish a strong foundation for advancing the theory and applications of starlike functions and their subclasses. Researchers in this field can look into this new subclass associated to the Daehee polynomial as seen below:

$$\mu'(z) \prec V(l, z)$$

$$\text{and } [\mu^{-1}(z)]' \prec V(l, \omega).$$

They can as well look into this new subclass too:

$$1 + \frac{z\mu''(z)}{\mu'(z)} \prec V(l, z)$$

$$\text{and } 1 + \frac{\omega[\mu^{-1}(z)]''}{[\mu^{-1}(z)]'} \prec V(l, \omega).$$

They can also use new operations to create some new subclasses.

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