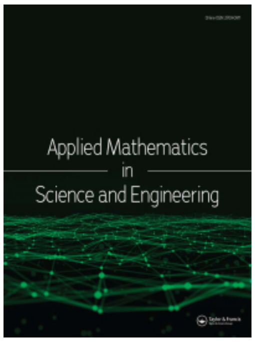

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Characterization of *p*-valent *q*-starlike functions through Hadamard product and Poisson distribution

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ABSTRACT
This paper presents significant advancement in the study of multivalent functions by introducing three new subclasses of p-valent q-starlike functions, defined with respect to higher-order q-derivatives within the open unit disk of the complex plane ($|z| < 1$). We utilized the Poisson distribution as a coefficient and applied the Hadmard product to derive several essential properties of these q-starlike functions, including distortion theorems and radius problems, particularly in case with negative coefficients. These findings enhance our understanding of the function behaviour and contribute to the broader mathematical discourse.

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1. Introduction and definitions

The q-calculus, a different version of classical calculus, has a different approach from the traditional one, which is limit-based calculus. It rather uses the parameter q to depict quantum properties, providing a discrete version of classical calculus. The initial parts of its development were systematically carried out by Jackson [1,2]. The recent history of the development of quantum group has allowed a broader application of q-calculus through focusing on its geometry and applications as well. More recent developments in quantum groups theory have also broadened the geometric perspective of q-analysis as one of main tools in the modern mathematics. It is worth noting that Aral and Gupta [3,4] carried out a considerable amount of mathematical work on the q-version of the Baskakov–Durrmeyer q-calculus for the beta function. Their contribution has expanded the concept of quantum counterparts of classical operators.

The q-calculus development has resulted significant advancements in operator theory. Innovations like the q-Picard and q-Gauss-Weierstrass integral operators, as discussed [5,6], have expanded the capabilities of traditional complex operators. These

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quantum adaptations offer valuable insights for analysing discrete quantum systems and their underlying mathematical frameworks. In a pivotal study by Wongsaijai and Sukantamala [7], q-calculus was utilized to generalize particular subclasses of starlike functions in Geometric Function Theory (GFT). Srivastava's extensive investigation of generalized q-hypergeometric functions [8] has further enriched the theoretical structure of q-calculus. This research has provided crucial tools for diverse applications in mathematical physics and special functions theory as mentioned in [5,6], establishing significant links between q-calculus and classical special functions. The study of analytic and bi-univalent functions has seen remarkable progress through the application of q-calculus, particularly in exploring their geometric properties and structural characteristics. Zhang et al. [9] utilized q-Hermite polynomials to investigate subclasses of such functions, while Shi et al. [10] examined their geometric behaviour through a generalized q-operator. Furthermore, Hu et al. [11] extended these ideas by employing the q-derivative operator with Gegenbauer polynomials, further enriching the theoretical framework and applications of q-fractional calculus in function theory.

Based on these foundational studies, Saliu et al. [12] have extensively explored various facets of q-calculus. Their studies have determined novel characteristics and uses of q-calculus, especially concerning functional analysis and operator theory, showing the adaptability and mathematical depth of this framework. The ongoing advances in q-calculus consistently underscore its importance across varied mathematical domains, ranging from quantum mechanics to function theory, solidifying its fundamental position in modern mathematical analysis. Its diverse applications encompass quantum groups, special functions and geometric function theory, emphasizing its role as a unifying element between discrete and continuous mathematics. Srivastava and Bansal undertook an in-depth analysis of the close-to-convex properties of q-Mittag–Leffler functions, as reported in [13]. In a related study, Srivastava et al. [14], see also [15], explored the behavioural characteristics of q-starlike functions within conic domains. The authors of [16] established noteworthy results regarding the upper bound for the third Hankel determinant. More recently, Mehmood, Srivastava, and their colleagues [17,18] conducted thorough investigations into various aspects of q-starlike functions, with a particular focus on their connections to Janowski functions. The authors successfully derived coefficient bounds for q-Lommel polynomials [19]. Moreover, the authors of [20,21] introduced innovative perspectives in the q-calculus, improving our understanding of q-polynomials and $(q - \lambda)$ differential operators.

In [22], Yang–Mills theories were quantized in a canonical framework. Finkelstein introduced a q-deformation of general theory of relativity, replacing the traditional Lorentz group with a q-Lorentz group [23]. The q-Yang–Mills equation was examined by Kamata and Nakamula in their study [24]. In 1990, the q-algebra $SU_q(2)$ was employed to investigate the rotational spectra of deformed nuclei [25]. It was demonstrated that $SU_q(2)$ effectively describes the spectra of superdeformed bands in nuclei as well as rotating bands with standard deformation [26]. Moreover, in 1992, researchers developed a q-deformed Aufbau principle for atoms and monoatomic ions, utilizing q-deformed chain $SO(4) > SO(3)_q$. Subsequently, as documented in [27], the researchers applied the q-algebra approach to investigate molecular backbending in silver hydride (AgH). Several years later in 1996, the superfluidity of ^4He was examined through a q-Heisenberg algebra framework [28]. Finkelstein established in 1967 that the dynamics of hydrogen atoms is linked to $O(3)$. The Wigner function D_{jmn} serves as a solution of this integral equation. Furthermore, the authors of [29] introduced a q-analogue of D_{jmn} using Ward numbers. In the realm of quantum theory, the momentum operator is represented as p_q, the wave function denoted by $\psi(x)$ and is given by

$$p_q \psi(x) = \frac{\hbar}{i} D_q \psi(x).$$

The substitution of a differential operator (D_q) with a difference operator implies the substitution of the conventional continuum with a fundamental lattice or, alternatively, by utilizing a space defined by non-commuting coordinates that give rise to this lattice structure [30]. The q-electroweak theory describes fundamental particles as solitons utilizing irreducible representations of $SU_q(2)$, as compared to [31]. Building upon foundational concepts in theoretical physics, Alain Connes developed a geometric framework that provided a deeper understanding of the Weinberg–Salam theory, specifically detailing the dynamics of electroweak interactions.

Having reviewed the historical development of q-calculus, we now present the fundamental definitions and concept details to our investigation. Let $\mathbb{C}$ be a complex plan and $\mathcal{T}(\mathbb{U})$ is the class of holomorphic functions within the open unit disk $\mathbb{U}$ given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Let $\mathcal{N}$ represent the subset of functions $Y(z) \in \mathcal{T}(\mathbb{U})$ then the Taylor–Maclaurin series expansion of $T(z)$ is given by

$$Y(z) = z + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (z \in \mathbb{U}), \tag{1}$$

while $Y(z)$ is known as p-valent function and satisfy the following normalization conditions:

$$Y(0) = Y'(0) - 1 = 0.$$

Also, $\mathcal{S} \subset A$ refers to the class of functions that are univalent in $\mathbb{U}$. The class of starlike functions in $\mathbb{U}$ is denoted by $\mathcal{S}^*$. It consists of normalized functions $Y(z) \in \mathbb{U}$ that satisfy the following conditions:

$$Y(z) \in \mathcal{S} \quad \text{and} \quad \Re\left(\frac{zY'(z)}{Y(z)}\right) > 0 \quad (\forall z \in \mathbb{U}). \tag{2}$$

Those functions which satisfy the above conditions are called startlike function.

Definition 1.1: Let $0 < q < 1$ and the q-number $[\Omega]_q$ is defined as follows:

$$[\Omega]_q := \begin{cases} \dfrac{1 - q^{\Omega}}{1 - q} & (\Omega \in \mathbb{C}) \\ \displaystyle\sum_{x=0}^{n-1} q^x = 1 + q + q^2 + \cdots + q^{n-1} & (\Omega = n \in \mathbb{N}). \end{cases}$$

Definition 1.2: Let $0 < q < 1$ and define the q-factorial $[\varepsilon]_q!$ by

$$[\varepsilon]_q! = \begin{cases} 1 & (\varepsilon = 0) \\ \prod_{u=1}^{a} [u]_q & (\varepsilon \in \mathbb{N}). \end{cases}$$

Definition 1.3: Let

$$g(z) = v_0 + v_1 z + v_2 z^2 + \cdots$$

and

$$h(z) = t_0 + t_1 z + t_2 z^2 + \cdots$$

the Hadamard product of the functions $g(z)$ and $h(z)$ denoted by

$$(h * g) = (g * h) = z + \sum_{k=2}^{\infty} v_k t_k z^k.$$

The idea of convolution or Hadamard product came from the integral

$$(g * h)(r^2 \, e^{i.\theta}) = \frac{1}{2\Pi} \int_0^{2\Pi} g(r \, e^{i(\theta - t)}) h(r \, e^{i\theta}) \, dt \quad \text{where } r < 1.$$

Definition 1.4: The distortion theorem stated that the class of any holomorphic function in the form of

$$h(z) = t_0 + t_1 z + t_2 z^2 + \cdots \quad \text{where } |z| < 1$$

and satisfy the following inequality:

$$\left| h(z) - \frac{1}{2\alpha} \right| \leq \frac{1}{2\alpha}, \quad 0 \leq \alpha < 1.$$

Definition 1.5 ([1]): The q-derivative of a function $Y(z)$ (1) is denoted by $(D_q Y)(z)$ is given by

$$\left(D_q Y\right)(z) = \begin{cases} \dfrac{Y(z) - Y\left(qz\right)}{z\left(1-q\right)} & (z \neq 0) \\ Y'(0) & (z = 0). \end{cases} \tag{3}$$

When $q \to 1-$, the q-derivative operator D_q becomes ordinary derivative D as given below

$$\lim_{q \to 1-} \left\{ \left(D_q Y\right)(z) \right\} = Y'(z).$$

Definition 1.6 ([32]): A function $Y(z) \in S$ is considered to be in class S_q^* of q-starlike functions in $\mathbb{U}$ if

$$Y(0) = Y'(0) - 1 = 0, \tag{4}$$

then

$$\left| \frac{z}{Y(z)} \left(D_q Y\right)(z) - \frac{1}{1-q} \right| \leq \frac{1}{1-q} \quad (z \in \mathbb{U}). \tag{5}$$

We readily observe that, as $q \to 1-$, the closed disk

$$\left| w - \frac{1}{1-q} \right| \leq \frac{1}{1-q}$$

$\mathcal{S}_q^*$ of q-starlike functions in $\mathbb{U}$ reduces to the known class $\mathcal{S}^*$ of normalized starlike functions with regard to the origin ($z = 0$).

In the following equations, we settled a power series so that its coefficients are probabilities of q-Poisson distribution as

$$Y(\psi, z) = z + \sum_{k=2}^{\infty} \frac{\psi_{k-1}}{[k-1]_q} e_q^{-\psi} z^k \quad (z \in \mathbb{U}), \tag{6}$$

by using Hadmard product we now introduce the following with

$$V(z) = z + \sum_{k=2}^{\infty} a_{k+p} z^k \quad (z \in \mathbb{U}),$$

$$\Psi_q(Y(\psi, z), V(z)) = Y(\psi, z) * V(z),$$

$$\Psi_q(Y(\psi, z), V(z)) = \left(z + \sum_{k=2}^{\infty} \frac{\psi_{k-1}}{[k-1]_q} e_q^{-\psi} z^k \right) * \left(z + \sum_{k=2}^{\infty} a_{k+p} z^k \right), \tag{7}$$

$$\Psi_q(Y(\psi, z), V(z), p) = z^p + \sum_{k=2}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q} e_q^{-\psi} a_{k+p} z^{k+p}.$$

The coefficients in (7) show the probabilities of Poisson distribution which was introduced by [33]. Harmonic functions are essential to geometric function theory. While harmonic functions satisfy the Laplace equation, this is strongly connected to Poisson equations. From the study of Poisson distribution theory, the Poisson equations arise. The Poisson distribution is a discrete probability distribution in probability theory and statistics. It calculates the probability of a specific number of events happening within a set time interval, assuming that these events occur at a constant average rate and independently of the time since the last event. Poisson distribution highly used in physics like astronomy, quantum physics and optics. Mathematically Poisson distribution can be defined as

$$f(m) = \frac{\nu^m}{m!} \mathrm{e}^{-\nu},$$

where e is Euler's number ($e = 2.71828\ldots$), m shows the number of occurrence and ν is the expected value. By taking q-derivative of (7) then

$$D_q^1 \Psi_q(Y(\psi, z), V(z)) = [p]_q z^{p-1} + \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q} e_q^{-\psi} [k+p]_q a_{k+p} z^{k+p-1}$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{matrix}$$

$$D_q^{(p)}\Psi_q(Y(\psi,z),V(z)) = [p]_q! + \sum_{k=1}^{\infty}\frac{[k+p]!\psi_{k+p-1}}{[k+p-1]_q[k]!}e_q^{-\psi}[k+p]a_{k+p}z^k$$

$$D_q^{(s)}\Psi_q(Y(\psi,z),V(z)) = \frac{[p]_q!}{[p-s]!}z^{p-s} + \sum_{k=1}^{\infty}\frac{[k+p]!\psi_{k+p-1}}{[n+p-s]_q[k+p-1]!}$$
$$e_q^{-\psi}[k+p]a_{k+p}z^{k+p-s}.$$

By using (6), we introduce three novel subclasses of $\mathcal{S}^*$ of q-starlike functions, characterized by the presence of s-order q-derivatives

$$\Lambda_q\mathcal{S}_1^*(v,\alpha,s), \quad \Lambda_q\mathcal{S}_2^*(v,\alpha,s) \quad \text{and} \quad \Lambda_q\mathcal{S}_3^*(v,\alpha,s).$$

Definition 1.7: A function $\Psi_q(Y(\psi,z),V(z)) \in \mathbb{U}$ will be in $\Lambda_q\mathcal{S}_1^*(v,\alpha,s)$ if the following condition meets:

$$\Re\left(\frac{z^v D_q^{(v+s)}\Psi_q(Y(\psi,z),V(z))}{D_q^{(s)}\Psi_q(Y(\psi,z),V(z))}\right) \le \alpha.$$

We designate $\Lambda_q\mathcal{S}_1^*(v,\alpha,s)$ as the class of multivalent (p-valent) q-starlike function of Type 1.

Definition 1.8: A function $\Psi_q(Y(\psi,z),V(z)) \in \mathbb{U}$ will be in $\Lambda_q\mathcal{S}_2^*(v,\alpha,s)$ if the following condition meets:

$$\left|\frac{\frac{z^v D_q^{(v+s)}\Psi_q(Y(\psi,z),V(z))}{D_q^{(s)}\Psi_q(Y(\psi,z),V(z))}-\alpha}{1-\alpha} - \frac{1}{1-q}\right| \le \frac{1}{1-q}.$$

We call $\Lambda_q\mathcal{S}_2^*(v,\alpha,s)$ the class of multivalent (p-valent) q-starlike function of Type 2.

Definition 1.9: A function $\Psi_q(Y(\psi,z),V(z) \in \mathbb{U}$ will be in $\Lambda_q\mathcal{S}_3^*(v,\alpha,s)$ if the following condition meets:

$$\left|\frac{z^v D_q^{(v+s)}\Psi_q(Y(\psi,z),V(z))}{D_q^{(s)}\Psi_q(Y(\psi,z),V(z))} - 1\right| \le 1-\alpha.$$

We call $\Lambda_q\mathcal{S}_3^*(v,\alpha,s)$ the class of multivalent (p-valent) q-starlike function of Type 3.

2. Main results

Theorem 2.1: *If* $0 < \alpha < 1$, *then*

$$\Lambda_q\mathcal{S}_3^*(v,\alpha,s) \subset \Lambda_q\mathcal{S}_2^*(v,\alpha,s) \subset \Lambda_q\mathcal{S}_1^*(v,\alpha,s).$$

Proof: By using definition

$$\left| \frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))} - 1 \right| \leq 1 - \alpha$$

$$\left| \frac{\frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi,z),V(z))}{D_q^{(s)} \Psi_q(Y(\psi,z),V(z))} - \alpha}{1-\alpha} \right| + 1 + \frac{q}{1-q} \leq 1 + \frac{q}{1-q} \tag{8}$$

$$\left| \frac{\frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi,z),V(z))}{D_q^{(s)} \Psi_q(Y(\psi,z),V(z))} - \alpha}{1-\alpha} - \frac{1}{1-q} \right| \leq \frac{1}{1-q}$$

The above inequality shows that $\Lambda_q \mathcal{S}_3^*(\nu, \alpha, s) \subset \Lambda_q \mathcal{S}_2^*(\nu, \alpha, s)$. Now by using Definition 1.8, we have

$$\left| \frac{\frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi,z),V(z))}{D_q^{(s)} \Psi_q(Y(\psi,z),V(z))} - \alpha}{1-\alpha} - \frac{1}{1-q} \right| \leq \frac{1}{1-q},$$

by simplification we can write as

$$\left| \frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))} - \frac{1-\alpha q}{1-q} \right| \leq \frac{1-\alpha}{1-q}. \tag{9}$$

In (9), we see that

$$\frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))}$$

lies on circle while $\frac{1-\alpha}{1-q}$ shows the radius and centre at $\frac{1-\alpha q}{1-q}$. We can also observe that

$$\frac{1-\alpha q}{1-q} - \frac{1-\alpha}{1-q} = \alpha,$$

which implies that

$$\Re\left(\frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))} \right) < \alpha. \tag{10}$$

The above inequality shows that $\Lambda_q \mathcal{S}_2^*(\nu, \alpha, s) \subset \Lambda_q \mathcal{S}_3^*(\nu, \alpha, s)$.

Finally using as coefficient inequality, we establish a sufficient condition for the class $\Lambda_q \mathcal{S}_m^*(\nu, \alpha, s)$ of generalized q-starlike function of Type 3, which also offers a corresponding sufficient condition for the class $\Lambda_q \mathcal{S}_1^*(\nu, \alpha, s), \Lambda_q \mathcal{S}_2^*(\nu, \alpha, s)$ Type 1 and Type 2, respectively. ■

Theorem 2.2: *A function $\Psi \in \mathcal{T}(\mathbb{U})$ of the form (7) is classified in the class $\Lambda_q \mathcal{S}_3^*(\nu, \alpha, s)$ if it meets the following coefficient inequality:*

$$\sum_{k=1}^{\infty} \frac{[k+p]!}{[k+p-1]_q!} \psi_{k+p-1}\, \mathrm{e}^{-\psi_q} \left((2-\alpha) \frac{[k+p]_q!}{[k+p-s]} - \frac{[k+p]_q!}{[k+p-\nu-s]_q!} \right) |a_{k+p}| < \frac{[p]_q!}{[p-\nu-s]_q!} - \alpha \frac{[p]_q!}{[p-s]_q!}. \tag{11}$$

Proof: Assuming (11) holds, it only remind to show that

$$\left| \frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))} - 1 \right| \leq 1 - \alpha.$$

Now

$$\begin{aligned} &\left| \frac{z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))} - 1 \right| \\ &= \left| \frac{D_q^{(s)} \Psi_q(Y(\psi, z), V(z)) - z^{\nu} D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))} \right| \\ &= \left| \frac{\frac{[p]_q!}{[p-s]_q!} + \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} \mathrm{e}^{-\psi_q} \frac{[k+p]_q!}{[k+p-s]_q!} |a_{k+p}|}{\frac{[p]_q!}{[p-s]_q!} - \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} \mathrm{e}^{-\psi_q} \frac{[k+p]_q!}{[k+p-s]_q!} |a_{k+p}|} \right. \\ &\quad \left. - \frac{\left(\frac{[p]_q!}{[p-\nu-s]_q!} + \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} \mathrm{e}^{-\psi_q} \frac{[k+p]_q!}{[k+p-\nu-s]_q!} |a_{k+p}| \right)}{\frac{[p]_q!}{[p-s]_q!} - \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} \mathrm{e}^{-\psi_q} \frac{[k+p]_q!}{[k+p-s]_q!} |a_{k+p}|} \right| \\ &\leq 1 - \alpha. \end{aligned}$$

After some simple calculations and by using triangular inequalities, we have the following:

$$\sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} \mathrm{e}^{-\psi_q} \left((2-\alpha) \frac{[k+p]_q!}{[k+p-s]_q!} - \frac{[k+p]_q!}{\left[k+p-\nu-s\right]_q!} \right) |a_{k+p}| \leq \frac{[p]_q!}{[p-\nu-s]_q!} - \alpha \frac{[p]_q!}{[p-s]_q!}$$

thus $\Psi \in \mathcal{T}(\mathbb{U})$ as required. ■

3. Analytic functions with negative coefficients

In this section, we will discuss a subclass of multivalent q-starlike function which contains negative coefficients. A subset of the class $\mathbb{U}$ that includes all such functions with negative coefficients, that is,

$$\Psi_q(Y(\psi, z), V(z), p) = z^p - \sum_{k=2}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q} e_q^{-\psi} a_{k+p} z^{k+p} \quad (z \in \mathbb{U}) \tag{12}$$

will be represented by the symbol T. We also let

$$T\Lambda_q \mathcal{S}_m^* (\nu, \alpha, s) := \Lambda_q \mathcal{S}_m^* (\nu, \alpha, s) \cap T \quad (\nu = 1, 2, 3).$$

Theorem 3.1: *If* $0 < \alpha < 1$, *then*

$$T\Lambda_q \mathcal{S}_1^* (\nu, \alpha, s) = T\Lambda_q \mathcal{S}_2^* (\nu, \alpha, s) = T\Lambda_q \mathcal{S}_3^* (\nu, \alpha, s).$$

Proof: By the virtue of Theorem 2.1, it suffices to show that

$$T\Lambda_q \mathcal{S}_1^* (\nu, \alpha, s) \subseteq T\Lambda_q \mathcal{S}_3^* (\nu, \alpha, s). \tag{13}$$

Indeed, in the light of Definition 1.7 for a function $\Psi \in T\Lambda_q \mathcal{S}_1^*(\nu, \alpha, s)$, we have

$$\Re\left(\frac{z^\nu D_q^{(\nu+s)} \Psi_q(Y(\psi, z), V(z))}{D_q^{(s)} \Psi_q(Y(\psi, z), V(z))}\right) \leq \alpha \tag{14}$$

$$\Re\left(\frac{z^\nu \left(\frac{[p]_q!}{[p-\nu-s]_q!} - \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} e^{-\psi_q} \frac{[k+p]_q!}{[k+p-\nu-s]_q!} \left|a_{k+p}\right| z^{k+p-\nu-s}\right)}{\frac{[p]_q!}{[p-s]_q!} - \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} e^{-\psi_q} \frac{[k+p]_q!}{[k+p-s]_q!} \left|a_{k+p}\right| z^{k+p-s})}\right) \leq \alpha. \tag{15}$$

If we choose z on real axis and let $z \to 1^-$, then

$$\frac{\frac{[p]_q!}{[p-\nu-s]_q!} - \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q} e^{-\psi_q} \frac{[k+p]_q!}{[k+p-\nu-s]_q} a_{k+p}}{\frac{[p]_q!}{[p-\nu]_q!} - \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]} e^{-\psi_q} \frac{[k+p]_q!}{[k+p-s]} a_{k+p}} \leq \alpha. \tag{16}$$

Let

$$\Omega = \frac{\psi_{k+p-1}}{\psi_{k+p-1}} - \alpha \frac{[k+p]_q!}{[k+p-\nu-s]_q},$$

then

$$\frac{[p]_q!}{[p-\nu-s]_q!} + \Omega \sum_{k=1}^{\infty} \frac{\psi_{k+p-1}}{[k+p-1]_q!} a_{k+p} \leq \alpha \frac{[p]_q!}{[p-s]_q!}.$$

The assertion of theorem indicates that types 1, 2 and 3 of generalized p-valently q-starlike functions are equivalent. For the purpose of clarification, we present the following distortion theorem using the notation $T\Lambda_q\mathcal{S}_m^*(\nu,\alpha,s)$ and $m = 1, 2, 3$. ■

Theorem 3.2: *If* $\Psi \in T\Lambda_q\mathcal{S}_m^*(\nu,\alpha,s)$ *and* $m \in \{1,2,3\}$, *then*

$$r^p - \left(\frac{\frac{[p]_q!}{[p-\nu-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!}}{\chi}\right) r^{p+1} \le |\Psi(z)| \le r^p + \left(\frac{\frac{[p]_q!}{[p-\nu-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!}}{\chi}\right) r^{p+1},$$

where $k \in \mathbb{N}$,

$$\chi = \frac{\psi_p}{[p]_q!}\,e^{-\psi_q}\left(\frac{[1+p]_q!}{[1+p-s]_q!} + (2-\alpha)\frac{[1+p]_q!}{[1+p-s]_q!}\right)$$

and

$$|z|^p = r^p \quad (0 < r < 1).$$

Proof: The following inequality is a direct consequence of Theorem 2.2:

$$\begin{aligned}
&\frac{\psi_p}{[p]_q!}\,e^{-\psi_q}\left(\frac{[1+p]_q!}{[1+p-s]_q!} + (2-\alpha)\frac{[1+p]_q!}{[1+p-s]_q!}\right)\sum_{k=1}^{\infty}|a_{k+p}| \\
&\le \sum_{k=1}^{\infty}\frac{\psi_{k+p-1}}{[k+p-1]_q!}\,e^{-\psi_q}\left((2-\alpha)\frac{[k+p]_q!}{[k+p-s]_q!} - \frac{[k+p]_q!}{[k+p-\nu-s]_q!}\right)|a_{k+p}| \\
&\le \frac{[p]_q!}{[p-\nu-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!}
\end{aligned}$$

in such way that

$$\begin{aligned}
|\Psi(z)| &\le r^p + \sum_{k=1}^{\infty}\left|t_{n+p}\right| r^{p+k} \le r^p + r^{p+1}\sum_{k=1}^{\infty}\left|a_{k+p}\right| \\
&\le r^p + \left(\frac{\frac{[p]_q!}{[p-\nu-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!}}{\frac{\psi_p}{[p]_q!}\,e^{-\psi_q}\left(\frac{[1+p]_q!}{[1+p-s]_q!} + (2-\alpha)\frac{[1+p]_q!}{[1+p-s]_q!}\right)}\right) r^{p+1}
\end{aligned}$$

likewise

$$\begin{aligned}
|\Psi(z)| &\ge r^p - \sum_{k=1}^{\infty}\left|a_{k+p}\right| r^{p+k} \ge r^p - r^{p+1}\sum_{k=1}^{\infty}\left|a_{k+p}\right| \\
&\ge r^p - \left(\frac{\frac{[p]_q!}{[p-\nu-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!}}{\frac{\psi_p}{[p]_q!}\,e^{-\psi_q}\left(\frac{[1+p]_q!}{[1+p-s]_q!} + (2-\alpha)\frac{[1+p]_q!}{[1+p-s]_q!}\right)}\right) r^{p+1}
\end{aligned}$$

we obtain with this the proof of Theorem 3.2 is complete. Theorem 3.3 can be proven using methods analogous to those applied in the proof of Theorem 3.2. Therefore, we will omit the detailed proof for Theorem 3.3. ■

Theorem 3.3: *If* $\Psi \in T\Lambda_q\mathcal{S}_m^*(v,\alpha,s)$ *and* $m \in \{1,2,3\}$, *then*

$$\begin{aligned} &pr^{p-1} - \left(\frac{\frac{[p]_q!}{[p-v-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!}}{\frac{\psi_p}{[p]_q!}\,\mathrm{e}^{-\psi_q}\left(\frac{[1+p]_q!}{[1+p-s]_q!} + (2-\alpha)\frac{[1+p]_q!}{[1+p-s]_q!}\right)}\right) r^p \\ &\leq |\Psi'(z)| \\ &\leq pr^{P-1} + \left(\frac{(p+1)(\frac{[p]_q!}{[p-v-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!})}{\frac{\psi_p}{[p]_q!}\,\mathrm{e}^{-\psi_q}\left(\frac{[1+p]_q!}{[1+p-s]_q!} + (2-\alpha)\frac{[1+p]_q!}{[1+p-s]_q!}\right)}\right) r^{p+1}, \end{aligned}$$

where $k \in \mathbb{N}$ *and* $|z|^p = r^p$ $(0 < r < 1)$. *Finally, we ascertain the radius of p-valent starlikeness for functions contained within the class* $T\Lambda_q\mathcal{S}_m^*(v,\alpha,s)$ *and* $m \in \{1,2,3\}$.

Theorem 3.4: *Let function* Ψ_q *given by* (12) *within the class* $\Lambda_q\mathcal{S}_m^*(v,\alpha,s)$ *where* $m = 1, 2, 3$ *if*

$$\inf_{n\geq 1}\left(\frac{p!\frac{\psi_{k+p-1}}{[k+p-1]_q!}\,\mathrm{e}^{-\psi_q}\left((2-\alpha)\frac{[k+p]_q!}{[k+p-s]_q!} - \frac{[k+p]_q!}{[k+p-v-s]_q!}\right)}{(k+p)!\frac{[p]_q!}{[p-v-s]_q!} - \alpha\frac{[p]_q!}{[p-s]_q!}}\right)^{\frac{1}{n}}$$

Proof: To prove Theorem 3.4, it is sufficient to show that

$$\left|\frac{\Psi_q^p(z)}{p!} - 1\right| < 1 \quad (|z| \leq r_0).$$

Now, we obtain

$$\begin{aligned} &\left|\frac{\Psi_q^p(z)}{p!} - 1\right| \\ &= \left|-\sum_{k=1}^{\infty}\frac{(k+p)!}{p!}\left|a_{k+p}\right| z^n\right| \\ &\leq \sum_{k=1}^{\infty}\frac{(k+p)!}{p!}\left|a_{k+p}\right| |z|^n. \end{aligned}$$

Thus

$$\left|\frac{\Psi_q^p(z)}{p!}-1\right|<1$$

and

$$\sum_{k=1}^{\infty}\frac{(k+p)!}{p!}\left|a_{k+p}\right||z|^n\leq 1. \tag{17}$$

In accordance with Theorem 2.2, inequality (17) holds if

$$\frac{(k+p)!}{p!}|z|^n\leq\frac{\frac{\psi_{k+p-1}}{[k+p-1]_q!}\,\mathrm{e}^{-\psi_q}\left((2-\alpha)\frac{[k+p]_q!}{[k+p-s]_q!}-\frac{[k+p]_q!}{[k+p-v-s]_q!}\right)}{\frac{[p]_q!}{[p-v-s]_q!}-\alpha\frac{[p]_q!}{[p-s]_q!}}$$

$$|z|^n\leq\frac{p!\frac{\psi_{k+p-1}}{[k+p-1]_q!}\,\mathrm{e}^{-\psi_q}\left((2-\alpha)\frac{[k+p]_q!}{[k+p-s]_q!}-\frac{[k+p]_q!}{[k+p-v-s]_q!}\right)}{(k+p)!\frac{[p]_q!}{[p-v-s]_q!}-\alpha\frac{[p]_q!}{[p-s]_q!}}$$

hence

$$|z|\leq\left(\frac{p!\frac{\psi_{k+p-1}}{[k+p-1]_q!}\,\mathrm{e}^{-\psi_q}\left((2-\alpha)\frac{[k+p]_q!}{[k+p-s]_q!}-\frac{[k+p]_q!}{[k+p-v-s]_q!}\right)}{(k+p)!\frac{[p]_q!}{[p-v-s]_q!}-\alpha\frac{[p]_q!}{[p-s]_q!}}\right)^{\frac{1}{n}}. \tag{18}$$

based on (18), Theorem 3.4 is now concluded. ■

4. Conclusion

In conclusion, this study defines new subclasses of q-starlike functions and comprehensively examine their properties and characteristics. By establishing relevant connections with earlier works, we provide a solid foundation for the validity and verification of our result, reinforcing the theoretical frame work surrounding these functions. The explore of basic or q-series and the basic hypergeometric functions, highlight their significance in various mathematical and applied context, such as approximation theory, complex analysis and geometric function theory.

Our finding not only enhances the understanding of the behaviour of these q-starlike function but also contributes to the development of distortion theorem and radius problems within this frame work. This work paves the way for further research, inviting scholars to delve deeper into the intricate properties of multivalent functions and their underlying mathematical structure. Overall, this research enriches the existing literature and opens new avenues for exploration in the field.

Author contributions

All authors contributed equally to writing this article. All authors read and approved the final manuscript.

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