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Research Article

Ideal-based quasi zero divisor graph

Ece Yetkin Çelikel*[1], Angsuman Das[2], Cihat Abdioğlu[3]

[1] Department of Electrical-Electronics Engineering, Faculty of Engineering, Hasan Kalyoncu University, Gaziantep, Turkey
[2] Department of Mathematics, Presidency University, Kolkata, India
[3] Department of Mathematics and Science Education, Faculty of Education, Karamanoğlu Mehmetbey University, Karaman, Turkey

Abstract

Let R be a commutative ring with identity and I a proper ideal of R. In this paper we introduce the ideal-based quasi zero divisor graph $Q\Gamma_I(R)$ of R with respect to I which is an undirected graph with vertex set $V = \{a \in R\backslash\sqrt{I} : ab \in I$ for some $b \in R\backslash\sqrt{I}\}$ and two distinct vertices a and b are adjacent if and only if $ab \in I$. We study the basic properties of this graph such as diameter, girth, dominaton number, etc. We also investigate the interplay between the ring theoretic properties of a Noetherian multiplication ring R and the graph-theoretic properties of $Q\Gamma_I(R)$.

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1. Introduction

The concept of zero divisor graph and studies on graph-theoretic properties of commutative rings were first initiated by Beck in [4]. However, in that paper he was mainly interested in colorings. Then, Anderson and Livingston [2] introduced and studied the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, whose vertices are the nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. Later on, the study on graphs associated with rings has attracted many researchs (see for instance [1], [3], [10] and [11]).

Now, let us recall some standard terminology and notations which will be used in this paper. Throughout, R will be a commutative ring with identity and as usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

Let I be a proper ideal of R. The *radical* of I, denoted by $\sqrt{I}$, is defined by $\{a \in R : a^n \in I$ for some positive integer $n\}$. In particular, the set of all nilpotent elements of R is denoted by $\sqrt{0}$. The ideal I of R is called *primary* if whenever $a, b \in R$ with $ab \in I$ and $a \notin I$ implies $b \in \sqrt{I}$, and called *prime* if $ab \in I$ and $a \notin I$ implies $b \in I$. In [6], Fuchs introduced and studied the concept of quasi-primary ideal. According to that paper, a proper ideal I is called *quasi-primary* if whenever $a, b \in R$ with $ab \in I$ and $a \notin \sqrt{I}$ implies

*Corresponding Author.
Email addresses: ece.celikel@hku.edu.tr; (E. Y. Çelikel), angsuman.maths@presiuniv.ac.in (A. Das), cabdioglu@kmu.edu.tr (C. Abdioğlu)
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$b \in \sqrt{I}$, or equivalently if $\sqrt{I}$ is prime. Clearly, every prime ideal is primary and every primary ideal is quasi-primary. It is also well-known that if I is a primary ideal, then $\sqrt{I}$ is a prime ideal. However, the converse of this relation does not hold in general. For instance, let R be a ring of all polynomials that coefficient of x is divisible by 3 with degree at most n for some positive integer n. Consider the ideal $I = (9x^2, 3x^3, x^4, x^5, x^6)$ of R. Then, $\sqrt{I} = (3x, x^2, x^3)$ is prime ideal, but I is not primary since $9x^2 \in I$ but neither $x^2 \in I$ nor $9 \in \sqrt{I}$. For undefined notions about ring theory, we refer the reader to [9].

Let $G = (V, E)$ be a graph, where $V = V(G)$ and $E = E(G)$ is the set of vertices and the set of edges, respectively. Then, G is called *connected* if there is a path between any two distinct vertices and is called *complete* if all vertices are adjacent. The complete graph on n vertices is denoted by K_n. The *clique number*, $\omega(G)$, is the greatest integer $n \geq 1$ such that $K_n \subseteq G$, and $\omega(G) = \infty$ if $K_n \subseteq G$ for all $n \geq 1$. The *distance* between two distinct vertices a and b, denoted by $d(a, b)$, is the length of a shortest path connecting a and b. If such a path does not exists, then we write $d(a, b) = \infty$. It is clear that $d(a, a) = 0$. The *diameter* of G will be denoted by $diam(G)$ and defined as $diam(G) = \sup\{d(a, b) : a$ and b are vertices of $G\}$. The *girth* of G, denoted by $gr(G)$, is defined as the length of the shortest cycle in G and $gr(G) = \infty$ if G has no cycle. A nonempty subset D of the vertex set $V(G)$ is called a *dominating set* if every vertex $V(G\backslash D)$ is adjacent to at least one vertex of D. The *domination number* $\gamma(G)$ is the minimum cardinality among the dominating sets of G. The *chromatic number* of G is defined as the minimal number of colors needed to color G and denoted by $\chi(G)$. We refer the reader to [5] for general background and undefined notions on graph theory.

In [12], Redmond defined the *ideal-based zero divisor graph*, $\Gamma_I(R)$, for a proper ideal I of R with vertices $\{x \in R\backslash I : xy \in I$ for some $y \in R\backslash I\}$, where two distinct vertices x and y are adjacent if and only if $xy \in I$. Quasi-primary ideals and ideal-based zero divisor graphs motivated us to define a new graph containing elements of $R\backslash\sqrt{I}$ as vertices.

The aim of this paper is to introduce and study some of the basic properties of the *ideal-based quasi zero divisor graph* $Q\Gamma_I(R)$ of a ring R which is an undirected graph with vertices $\{a \in R\backslash\sqrt{I} : ab \in I$ for some $b \in R\backslash\sqrt{I}\}$ where I is a proper ideal of R and two distinct vertices a and b are adjacent if and only if $ab \in I$. Throughout the study we write $a \backsim b$ whenever the vertices a and b are adjacent.

In Section 2, we start with some trivial relations and some examples showing that under which conditions $Q\Gamma_I(R)$ and $\Gamma_I(R)$ coincides. We also investigate the graph properties of $Q\Gamma_I(R)$ such as diameter, girth, chromatic number, etc. In Theorem 2.9 the relationship between $Q\Gamma_I(R)$ and $Q\Gamma_I(R/I)$ is investigated. Among many other results in this section it is shown that $Q\Gamma_I(R)$ has no cut-vertex (Theroem 2.18).

In Section 3, we study ideal-based quasi zero divisor graphs of Noetherian multiplication rings. Especially, we investigate clique and chromatic numbers besides the diameter and the girth of the graph $Q\Gamma_I(R)$ for a Noetherian multiplication ring. In particular, the ideal-based quasi zero divisor graph of $\mathbb{Z}_m$ is entirely characterized. Moreover, we conclude the characterization for $Q\Gamma_I(R)$ (Theorem 3.2).

2. Basic properties of ideal-based quasi zero divisor graph

We start this section with an example to demonstrate the structure of $Q\Gamma_I(R)$ and the relationship between $Q\Gamma_I(R)$, $\Gamma_I(R)$ and $\Gamma(R)$.

Example 2.1. (1) Let $R = \mathbb{Z}_6$ and $I = 0$. Then, $Q\Gamma_I(R)$, $\Gamma_I(R)$ and $\Gamma(R)$ coincide.

(2) Let $R = \mathbb{Z}_{12}$ and $I = 0$. Then, $Q\Gamma_I(R)$ and $\Gamma_I(R)$ are different graphs as shown below. Moreover, this example denies the probable idea that the graph $Q\Gamma_I(R)$ arise by taking radical of an ideal in ideal-based zero divisor graph.

Figure 1. $Q\Gamma_0(\mathbb{Z}_6)$, $\Gamma_0(\mathbb{Z}_6)$, $\Gamma(\mathbb{Z}_6)$

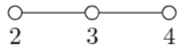
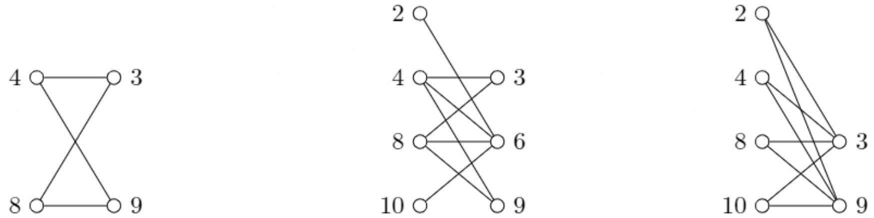


Figure 2. $Q\Gamma_{(0)}(\mathbb{Z}_{12})$ (left) and $\Gamma_{(0)}(\mathbb{Z}_{12})$ (centre) and $\Gamma_{\sqrt{0}}(\mathbb{Z}_{12})$ (right)



To see the general case for $\mathbb{Z}_n$ please see the Corollaries 3.7 and 3.8.

Proposition 2.2. *Let R be a ring and I a proper ideal of R.*

(1) *If R/I is a reduced ring (or equivalently, if $\sqrt{I} = I$), then the ideal-based quasi zero divisor graph and the ideal-based zero divisor graph coincide.*
(2) *I is a quasi primary ideal of R if and only if $Q\Gamma_I(R) = \emptyset$.*

Proof. Clear by definitions. □

Proposition 2.3. *Let R be a ring and I a proper ideal of R.*

(1) *$Q\Gamma_I(R)$ is an induced subgraph of $\Gamma_I(R)$.*
(2) *$Q\Gamma_I(R)$ is a subgraph of $\Gamma_{\sqrt{I}}(R)$.*

Proof. (1) Let $a \backsim b$ in $Q\Gamma_I(R)$. Then $ab \in I$ for $b \in R\backslash\sqrt{I}$ and so $ab \in I$ for $b \in R\backslash I$. Hence, $a \backsim b$ in $\Gamma_I(R)$.
(2) This part is clear as $ab \in I$ implies $ab \in \sqrt{I}$.

□

The following example shows that $Q\Gamma_I(R)$ need not to be an induced subgraph of $\Gamma_{\sqrt{I}}(R)$.

Example 2.4. Let $R = \mathbb{Z}_{60}$ and $I = 0$. Then, it is easy to see that the vertices 10 and 15 are adjacent in $\Gamma_{\sqrt{I}}(R)$ but not adjacent in $Q\Gamma_I(R)$. So, $Q\Gamma_I(R)$ is not an induced subgraph of $\Gamma_{\sqrt{I}}(R)$.

In Example 2.4, observe that $\sqrt{I} \neq I$ and $Q\Gamma_I(R)$ is not an induced subgraph. But, $\sqrt{I} \neq I$ does not mean that $Q\Gamma_I(R)$ is not an induced subgraph (see the graphs left and right in Figure 2).

Lemma 2.5. *Let R be a ring and I a nonzero proper ideal of R. Then $Q\Gamma_I(R)$ cannot be complete, i.e., $diam(Q\Gamma_I(R)) > 1$.*

Proof. Assume that $diam(Q\Gamma_I(R)) = 1$. Suppose that x is a vertex of $Q\Gamma_I(R)$. It is clear that $x + i \neq x$ is also a vertex of $Q\Gamma_I(R)$, where $0 \neq i \in I$. Hence $x(x + i) \in I$ implies $x^2 \in I$, a contradiction. Thus, $diam(Q\Gamma_I(R)) > 1$. □

Note that in Lemma 2.5, the condition $I \neq 0$ is not superficial. For instance, put $p = 2$ in Example 2.17. Then, $Q\Gamma_0(\mathbb{Z}_2 \times \mathbb{Z}_2)$ is complete with the only adjacent vertices $(1, 0)$ and $(0, 1)$.

Theorem 2.6. *Let I be a proper ideal of R. Then $Q\Gamma_I(R)$ is a connected graph with $diam(Q\Gamma_I(R)) \leq 3$.*

Proof. Let a and b are distinct vertices of $Q\Gamma_I(R)$. If $ab \in I$, then $a \backsim b$, so $d(a,b) = 1$. Suppose that $ab \notin I$. Then there exist $c, d \in R \backslash \sqrt{I}$ such that $ac \in I$ and $bd \in I$. If $c = d$, then $a \backsim c \backsim b$, so $d(a,b) = 2$. Assume that $c \neq d$. Then we have the following cases:
Case I. If $cd \notin \sqrt{I}$, then $a \backsim cd \backsim b$, so $d(a,b) = 2$.
Case II. If $cd \in \sqrt{I} - I$, then there exists an integer $n \geq 2$ such that $(cd)^n \in I$. Hence $a \backsim c^n \backsim d^n \backsim b$, so $d(a,b) = 3$.
Case III. If $cd \in I$, then $a \backsim c \backsim d \backsim b$, so $d(a,b) = 3$.
Thus $Q\Gamma_I(R)$ is connected and $diam(Q\Gamma_I(R)) \leq 3$. □

Theorem 2.7. *Let I be a proper ideal of R. If $Q\Gamma_I(R)$ contains a cycle, then $gr(Q\Gamma_I(R)) \leq 4$.*

Proof. Assume that $Q\Gamma_I(R)$ contains a cycle $a_0 \backsim a_1 \backsim \cdots \backsim a_n \backsim a_0$ such that $a_i a_j \notin I$ in case $j \neq i+1$ for all $i, j \in \{0, 1, ..., n\}$. Here we have two cases: $a_1 a_{n-1} \notin \sqrt{I}$ or $a_1 a_{n-1} \in \sqrt{I}$.
Case I: Assume that $a_1 a_{n-1} \notin \sqrt{I}$. Then, we have $a_0 \backsim a_1 a_{n-1} \backsim a_n$. Here, if $a_1 a_{n-1} = a_0$ then $a_0^2 \in I$, i.e. $a_0 \in \sqrt{I}$, a contradiction. Similarly, one can see that $a_1 a_{n-1} \neq a_n$. Hence, $a_0 \backsim a_1 a_{n-1} \backsim a_n \backsim a_0$ is a 3−cycle.
Case II: Assume that $a_1 a_{n-1} \in \sqrt{I}$. Then there exists the least positive integer $k \geq 2$ such that $(a_1 a_{n-1})^k \in I$. Hence $a_0 \backsim a_1^k \backsim a_{n-1}^k \backsim a_n \backsim a_0$ is a 4−cycle. □

Thus $gr(Q\Gamma_I(R)) \leq 4$.

Theorem 2.8. *Let R be a ring and I a proper ideal of R which is not quasi primary. Then $gr(Q\Gamma_{I[x]}(R[x])) \leq 4$.*

Proof. Since I is not quasi primary, there exist $a, b \in R \backslash \sqrt{I}$ such that $ab \in I$. Hence, $a \backsim b \backsim ax \backsim bx \backsim a$ is a 4−cycle. Thus, $gr(Q\Gamma_{I[x]}(R[x])) \leq 4$. □

In the next theorem, we give a relationship between $Q\Gamma_I(R)$ and $Q\Gamma_0(R/I)$.

Theorem 2.9. *Let I be a proper ideal of R and $a, b \in R \setminus \sqrt{I}$.*

(1) *a is adjacent to b in $Q\Gamma_I(R)$ if and only if $a+I$ is adjacent to $b+I$ in $Q\Gamma_0(R/I)$.*
(2) *$diam(Q\Gamma_I(R)) = diam(Q\Gamma_0(R/I))$ and $gr(Q\Gamma_I(R)) = gr(Q\Gamma_0(R/I))$.*

Proof.
(1) It is to be noted that $a \in V(Q\Gamma_I(R))$ if and only if $a + I \in V(Q\Gamma_0(R/I))$. Now $a \sim b$ in $Q\Gamma_I(R) \Leftrightarrow ab \in I \Leftrightarrow (a+I)(b+I) = I \Leftrightarrow a+I \sim b+I$ in $Q\Gamma_0(R/I)$.
At this point, we should be careful about the case when $a \sim b$ in $Q\Gamma_I(R)$ but $a + I = b + I$, because if this happens then the claim fails. However, we will show that this situation does not happen. For, if $a \sim b$ in $Q\Gamma_I(R)$ and $a + I = b + I$, then we have $ab, a - b \in I$. This implies $a^2 - ab = a(a-b) \in I$ and hence $a^2 \in I$, i.e., $a \in \sqrt{I}$, a contradiction.
(2) From part (1), it is clear that $d(a,b) = 1$ in $Q\Gamma_I(R)$ if and only if $d(a+I, b+I) = 1$ in $Q\Gamma_0(R/I)$. Now, $d(a,b) = 2$ in $Q\Gamma_I(R)$ if and only if $ab \notin I$ and there exists $c \in R \setminus \sqrt{I}$ such that $ac, bc \in I$ if and only if $d(a+I, b+I) = 2$ in $Q\Gamma_0(R/I)$. Similarly, $d(a,b) = 3$ in $Q\Gamma_I(R)$ if and only if $ab \notin I$ and there exists $c \in R \setminus \sqrt{I}$ such that $ac, bc \in I$ and there exist $c_1, c_2 \in R \setminus \sqrt{I}$ such that $ac_1, c_1c_2, bc_2 \in I$ if and only if $d(a+I, b+I) = 3$ in $Q\Gamma_0(R/I)$.
From Theorem 2.6, as diameter of any ideal-based quasi zero divisor graph is less than or equal to 3, we have $diam(Q\Gamma_I(R)) = diam(Q\Gamma_0(R/I))$ and $gr(Q\Gamma_I(R)) = gr(Q\Gamma_0(R/I))$.

□

A graph H is called a *retract* of G if there are homomorphisms $\rho : G \to H$ and $\varphi : H \to G$ such that $\rho \circ \varphi = id_H$. The homomorphism ρ is called a *retraction* (see [8, Definition 2.16]).

Proposition 2.10. [8, Observation 2.17] If H is a retract of G, then chromatic number and clique number of G and H are same.

Theorem 2.11. *$Q\Gamma_0(R/I)$ is a retract of $Q\Gamma_I(R)$.*

Proof. Define a map $\rho : V(Q\Gamma_I(R)) \to V(Q\Gamma_0(R/I))$ by $\rho(x) = x + I$. Again, for each coset $x + I \in V(Q\Gamma_0(R/I))$, choose and fix a representative $x^* \in x + I$ and define $\varphi : V(Q\Gamma_0(R/I)) \to V(Q\Gamma_I(R))$ by $\varphi(x + I) = x^*$. It is clear from Theorem 2.9 part (1) that ρ is a surjective graph homomorphism and φ is a graph homomorphism.

Moreover, $\rho \circ \varphi : V(Q\Gamma_0(R/I)) \to V(Q\Gamma_I(R))$ is given by $\rho \circ \varphi(x + I) = \rho(x^*) = x^* + I = x + I$, i.e., $\rho \circ \varphi$ is the identity map on $Q\Gamma_0(R/I)$. Thus $Q\Gamma_0(R/I)$ is a retract of $Q\Gamma_I(R)$. □

Corollary 2.12. *$Q\Gamma_0(R/I)$ and $Q\Gamma_I(R)$ have same chromatic number and clique number.*

Proof. It follows from Proposition 2.10 and Theorem 2.11. □

Theorem 2.13. *Let I be a proper ideal of R and $a,\ b \in R\backslash\sqrt{I}$. Then the following statements hold:*

(1) *If $a + I$ is adjacent to $b + I$ in $\Gamma(R/I)$, then a is adjacent to b in $Q\Gamma_I(R)$.*
(2) *If a is adjacent to b in $Q\Gamma_I(R)$, then $a+\sqrt{I}$ and $b+\sqrt{I}$ are always distinct elements, and also they are adjacent in $\Gamma(R/\sqrt{I})$. Furthermore, $Q\Gamma_I(R)$ is isomorphic to a subgraph of $\Gamma(R/\sqrt{I})$.*

Proof. (1) Suppose that $a + I \backsim b + I$ in $\Gamma(R/I)$. Hence $(a+I)(b+I) = 0+I$, so $ab \in I$. Since our assumption is $a,\ b \in R\backslash\sqrt{I}$, we have $a \backsim b$ in $Q\Gamma_I(R)$.

(2) Suppose that $a \backsim b$ in $Q\Gamma_I(R)$ and assume on the contrary that $a + \sqrt{I} = b + \sqrt{I}$. Then $ab \in I$ and $a - b \in \sqrt{I}$. Hence $a(a - b) \in \sqrt{I}$, it follows $a^2 \in \sqrt{I}$. Thus $a \in \sqrt{I}$, a contradiction. Consequently, $a + \sqrt{I} \neq b + \sqrt{I}$. Now, since $ab \in I$ and $a, b \in R\backslash\sqrt{I}$, $(a + \sqrt{I})(b + \sqrt{I}) = 0 + \sqrt{I}$. It means $a + \sqrt{I} \backsim b + \sqrt{I}$ in $\Gamma(R/\sqrt{I})$.

Suppose that the vertices of $\Gamma(R/\sqrt{I})$ is $\{a_i+\sqrt{I} : a_i \notin \sqrt{I}\}$. Now, we show that $Q\Gamma_I(R)$ is isomorphic to a subgraph of $\Gamma(R/\sqrt{I})$. We define a graph G with vertices $\{a_i : a_i + \sqrt{I}$ is a vertex of $\Gamma(R/\sqrt{I})\}$ where $a_i \backsim a_j$ if whenever $a_i a_j \in I$. Then G is a subgraph of $\Gamma(R/\sqrt{I})$. □

The next remark gives a method to construct $Q\Gamma_I(R)$ from $\Gamma(R/\sqrt{I})$.

Remark 2.14. Let I be an ideal of a ring R. We construct the graph $Q\Gamma_I(R)$ as the following method: Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a set of coset representatives of the vertices of $\Gamma(R/\sqrt{I})$. We define a graph G with vertices $\{a_i : a_i + \sqrt{I}$ is a vertex of $\Gamma(R/\sqrt{I})\}$. If $a_i a_j \notin I$, then omit these vertices. Hence $a_i \backsim a_j$ whenever $a_i a_j \in I$. Then G is a subgraph of $\Gamma(R/\sqrt{I})$.

Note that $\omega(Q\Gamma_I(R)) \leq \omega(\Gamma(R/\sqrt{I}))$ since $Q\Gamma_I(R)$ is isomorphic to a subgraph of $\Gamma(R/\sqrt{I})$.

Theorem 2.15. *Let I be a proper ideal of a ring R. If there exists a vertex of $Q\Gamma_I(R)$ which is adjacent to every other vertex of $Q\Gamma_I(R)$, then $I = 0$.*

Proof. Suppose that $a \in Q\Gamma_I(R)$ is adjacent to every other vertex of $Q\Gamma_I(R)$ and $I \neq 0$. Then there exists $0 \neq b \in I$. Observe that $a \neq a + b \in R\backslash\sqrt{I}$ and $a + b$ is also a vertex which is adjacent to every other vertex of $Q\Gamma_I(R)$. Hence $a(a + b) \in I$; and so we have $a^2 \in I$, a conradiction. Thus $I = 0$. □

The following example shows that the converse of Theorem 2.15 is not true in general.

Example 2.16. Let $R = \mathbb{Z}_{60}$ and $I = 0$. Then there is no vertex in $Q\Gamma_0(\mathbb{Z}_{60})$ which is adjacent to every other vertex in this graph. Indeed, $4, 5 \in Q\Gamma_0(\mathbb{Z}_{60})$ and $d(4, 5) = 3$. (one of the path is $4 \backsim 15 \backsim 12 \backsim 5$)

Example 2.17. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_p$ and $I = (0, 0)$, where $n \geq 2$. Then, it is clear that the vertex $(1, 0)$ is adjacent to $(0, 1), (0, 2), \ldots, (0, p-1)$.

Recall that a vertex a of a connected graph G is said to be a *cut-vertex* of G if there exist vertices x and y of G such that a is in every path from x to y where x, y and a are distinct.

Theorem 2.18. *Let I be a nonzero proper ideal of R. Then $Q\Gamma_I(R)$ has no cut-vertex.*

Proof. Suppose that a is a cut-vertex of $Q\Gamma_I(R)$. Then there exist vertices x,$y \in R\backslash\sqrt{I}$ such that a lies on every path from x to y. Since $diam(Q\Gamma_I(R)) \leq 3$, the shortest path from x to y is of the length 2 or 3.
Case I: Suppose that $x \backsim a \backsim y$ is a path of the shortest lenght from x to y. Hence $x+\sqrt{I} \neq a+\sqrt{I}$ and $y+\sqrt{I} \neq a+\sqrt{I}$ by Theorem 2.13. Let $0 \neq i \in I$. Since $x(a+i) \in I$ and $y(a+i) \in I$, we conclude that $x \backsim (a+i) \backsim y$ is a path in $Q\Gamma_I(R)$, a contradiction.
Case II: Suppose that $x \backsim a \backsim b \backsim y$ is a path of the shortest lenght from x to y. Hence $a+\sqrt{I} \neq b+\sqrt{I}$ by Theorem 2.13. Let $0 \neq i \in I$. Since $x(a+i) \in I$ and $b(a+i) \in I$, we conclude that $x \backsim (a+i) \backsim b \backsim y$ is a path in $Q\Gamma_I(R)$, a contradiction.
Thus $Q\Gamma_I(R)$ has no cut-vertex. □

3. Ideal-based quasi zero divisor graph of a Noetherian multiplication ring

Recall that a ring R is called a *multiplication ring* if whenever I, J are ideals of R with $I \subseteq J$, then there exists an ideal K of R such that $I = JK$. The aim of this section is to characterize ideal-based quasi zero divisor graphs of Noetherian multiplication rings. For this purpose, we need the following lemma.

Lemma 3.1. *Let R be a ring with identity. Then, the following are equivalent:*

(1) *R is a Noetherian multiplication ring.*
(2) *Each primary ideal of R is a prime power, i.e., if Q is a primary ideal of R, then $Q = P^n$ for some P prime ideal of R and $n \geq 0$.*

Proof. The result is clear from [7, 39.4 Proposition] and [7, Exercise 9 in S. 39]. □

Throughout, R will be a Noetherian multiplication ring. Note that Dedekind Domains are particular examples of Noetherian multiplication ring. Thus all results in this section is also valid for Dedekind Domains.

Theorem 3.2. *Let I be a proper ideal of R. Then, one of the following statements holds:*

(1) $Q\Gamma_I(R) = \emptyset$.
(2) $Q\Gamma_I(R)$ *is a complete bipartite graph.*
(3) $Q\Gamma_I(R)$ *is a k-partite graph for $k \geq 3$.*

Proof. Suppose that $Q\Gamma_I(R) \neq \emptyset$. Since R is Noetherian, I has a primary decomposition. Then, $I = Q_1 \cap \cdots \cap Q_k$ where Q_i $(i = 1, ..., k)$ are primary ideals of R. From Lemma 3.1, $Q_i = P_i^{\alpha_i}$ for some prime ideal P_i of R and $\alpha_i \geq 1$. Hence $I = P_1^{\alpha_1} \cap \cdots \cap P_k^{\alpha_k}$.

Case I. If $k = 1$, then $Q\Gamma_I(R) = \emptyset$ by Proposition 2.2 (2).

Case II. Let $k = 2$. Then, $I = P_1^{\alpha_1} \cap P_2^{\alpha_2}$ where P_1, P_2 are distinct primes. Hence the vertex set of the graph $V = (P_1^{\alpha_1} \cup P_2^{\alpha_2})\backslash(P_1 \cap P_2)$. Put $V_1 = P_2^{\alpha_2}\backslash P_1$ and $V_2 = P_1^{\alpha_1}\backslash P_2$. Note that in this case $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$. Moreover, V_1, V_2 are independent sets and any vertex in V_1 is adjacent to any arbitrary vertex in V_2. Thus, $Q\Gamma_I(R)$ is a complete bipartite graph.

Case III. Suppose that $k \geq 3$. We construct the vertex set V of $Q\Gamma_I(R)$ and partitions as follows:

$$V = \left(\bigcup_{i=1}^{k} P_i^{\alpha_i}\right) \setminus \left(\bigcap_{i=1}^{k} P_i\right)$$

and define $V_i = V \setminus P_i$ for $i = 1, 2, \ldots, k$. We claim that $V = \bigcup_{i=1}^{k} V_i$. Suppose there exists $x \in V \setminus \bigcup_{i=1}^{k} V_i$, then $x \in \bigcap_{i=1}^{k} V_i^c = \bigcap_{i=1}^{k} P_i$, a contradiction as $x \in V$. Thus $V = \bigcup_{i=1}^{k} V_i$. Clearly V_i's are independent sets. But V_i's are not pairwise disjoint. However, consider the sets recursively

$$W_1 = V_1; W_2 = V_2 \setminus V_1; W_3 = V_3 \setminus (V_1 \cup V_2), \ldots, W_k = V_k \setminus \left(\bigcup_{i=1}^{k-1} V_i\right).$$

It can be checked that W_i's are disjoint independent sets with $\bigcup_{i=1}^{k} W_i = V$. Thus $Q\Gamma_I(R)$ is k-partite. □

Corollary 3.3. *Let $I = P_1^{\alpha_1} \cap \cdots \cap P_k^{\alpha_k}$ where P_i's are distinct prime ideals of R and $k > 1$. Then the clique number ω of $Q\Gamma_I(R)$ is k.*

Proof. From Theorem 3.2, we have that $Q\Gamma_I(R)$ is k-partite. We claim that $\omega \leq k$. If not, let $\omega \geq k+1$. Then, by pigeon-hole principle, there exist at least two vertices a and b from the same partite set in any clique. However, as partite sets are independent, we arrive at a contradiction. Thus $\omega \leq k$. Now, for each $i = 1, 2, \ldots, k$, choose an element $x_i \in \bigcap_{\substack{t=1\\t\neq i}}^{k} P_t^{\alpha_t}$. Clearly x_i's belong to $V(Q\Gamma_I(R))$. Moreover, x_i is adjacent to x_j in $Q\Gamma_I(R)$ for $i \neq j$. Thus we get a clique of size k. Hence the corollary follows. □

Corollary 3.4. *Let $I = P_1^{\alpha_1} \cap \cdots \cap P_k^{\alpha_k}$ where P_i's are distinct prime ideals of R and $k > 1$. Then, $\chi(Q\Gamma_I(R)) = k$.*

Proof. Since $Q\Gamma_I(R)$ is k-partite, we have $\chi \leq k$. Again, as $\omega = k$, we have $\chi \geq k$. Thus the corollary follows. □

Theorem 3.5. *Let $I = P_1^{\alpha_1} \cap \cdots \cap P_k^{\alpha_k}$ where P_i's are distinct prime ideals of R and $k > 1$. Then, diameter and girth of $Q\Gamma_I(R)$ is given by*

$$diam(Q\Gamma_I(R)) = \begin{cases} 2, & \text{if } k = 2 \\ 3, & \text{if } k > 2 \end{cases} \quad \text{and} \quad gr(Q\Gamma_I(R)) = \begin{cases} 4, & \text{if } k = 2 \\ 3, & \text{if } k > 2 \end{cases}.$$

Proof. If $I = P_1{}^{\alpha_1} \cap P_2{}^{\alpha_2}$, then by Theorem 3.2, $Q\Gamma_I(R)$ has diameter 2 and girth 4.

If there are more than two distinct prime ideals containing I, then let P_1, P_2, P_3 be three distinct prime ideals of R. Consider the vertices $u \in P_1{}^{\alpha_1}$ and $v \in P_2{}^{\alpha_2}$. Clearly they are not adjacent. If possible, let a be a common neighbour of u and v. Then, $au, av \in I$ and hence $a \in \bigcap_{j=2}^{k} P_j^{\alpha_j}$ and $a \in \bigcap_{\substack{j=1\\j\neq 2}}^{k} P_j^{\alpha_j}$, i.e., $a \in \bigcap_{j=1}^{k} P_j$. However, this contradicts that $a \in V(Q\Gamma_I(R))$. Hence $d(u, v) > 2$. Now, by Theorem 2.6, we know that $diam(Q\Gamma_I(R)) \leq 3$. Thus $diam(Q\Gamma_I(R)) = 3$.

Again, consider $a \in \bigcap_{j=2}^{k} P_j^{\alpha_j}$, $b \in \bigcap_{\substack{j=1 \\ j\neq 2}}^{k} P_j^{\alpha_j}$, $c \in \bigcap_{\substack{j=1 \\ j\neq 3}}^{k} P_j^{\alpha_j}$. Clearly $a, b, c \in V(Q\Gamma_I(R))$ and they form a triangle. Hence $gr(Q\Gamma_I(R)) = 3$ and the theorem follows. □

Let $R = \mathbb{Z}$. Then, any ideal of R is of the form $m\mathbb{Z}$. We conclude the following characterizations for ideal-based quasi zero divisor graph of $\mathbb{Z}$ by the next Theorem and Corollaries:

Theorem 3.6. *Let $m = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ where p_i's are distinct primes and $k > 1$. Then domination number γ of $Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})$ is k.*

Proof. For $i = 1, 2, \ldots, k$, consider the vertices $x_i = m/{p_i}^{\alpha_i}$. We claim that $S = \{x_i : i = 1, 2, \ldots, k\}$ is a dominating set for $Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})$. Let x be an arbitrary vertex in $Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})$. Then $p_1p_2\cdots p_k$ does not divide x and there exists $j \in \{1, 2, \ldots, k\}$ such that ${p_j}^{\alpha_j}$ divide x. Observe that $xx_j \in m\mathbb{Z}$, i.e., x is adjacent to x_j. Thus S is a dominating set and hence $\gamma \leq k$.

If possible, let $\gamma < k$. Then there exists a dominating set S' with $k-1$ vertices. Let $S' = \{y_1, y_2, \ldots, y_{k-1}\}$. Consider the set of vertices $D = \{{p_1}^{\alpha_1}, {p_2}^{\alpha_2}, \ldots, {p_k}^{\alpha_k}\}$. If any ${p_i}^{\alpha_i} \in S'$, then we replace ${p_i}^{\alpha_i}$ in D by $p{p_i}^{\alpha_i}$ where p is a prime which does not divide m and $p{p_i}^{\alpha_i} \notin S'$. This can be guaranteed as choice of such a p is infinite. Thus $D \cap S' = \emptyset$. Since S' is a dominating set, each element of D is adjacent to some element of S'. We claim that two distinct elements of ${p_i}^{\alpha_i}$ and ${p_j}^{\alpha_j}$ of D can not be dominated by same y_t. Because, if it happens then ${p_i}^{\alpha_i}y_t, {p_j}^{\alpha_j}y_t \in m\mathbb{Z}$, i.e., both $m/{p_i}^{\alpha_i}$ and $m/{p_j}^{\alpha_j}$ divides y_t, i.e., their l.c.m. divides y_t, i.e., $m|y_t$, i.e., $y_t \in m\mathbb{Z}$, a contradiction. Therefore distinct ${p_i}^{\alpha_i}$'s are dominated by distinct elements of S' and hence S' should contain at least k vertices, a contradiction. Thus $\gamma = k$ and the theorem holds. □

Corollary 3.7. *Let $I = m\mathbb{Z}$ be an ideal of $\mathbb{Z}$. Then,*

(1) *If $m = 0$ or $m = p^k$ where p is prime and k is a positive integer, then $Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})$ is a null graph.*
(2) *If $m = {p_1}^{\alpha_1}{p_2}^{\alpha_2}$ where p_1, p_2 are distinct primes, then $Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})$ is a complete bipartite graph with $diam(Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = 2$ and $gr(Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = 4$.*
(3) *If $m = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ where p_i's are distinct primes and $k > 2$, then $Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})$ is a k-partite graph with $diam(Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = gr(Q\Gamma_{m\mathbb{Z}}(\mathbb{Z})) = 3$, clique number $\omega = k$, chromatic number $\chi = k$ and the domination number $\gamma = k$.*

As an application of Theorem 2.9, Theorem 3.5 and Theorem 3.6, we conclude the following result for $\mathbb{Z}_m$ with respect to the the zero ideal.

Corollary 3.8. *Let $m = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\cdots{p_k}^{\alpha_k}$ where p_i's are distinct primes and $k > 1$. Then,*

(1) *the diameter and girth of $Q\Gamma_0(\mathbb{Z}_m)$ are given by*

$$diam(Q\Gamma_0(\mathbb{Z}_m)) = \begin{cases} 2, & \text{if } k = 2 \\ 3, & \text{if } k > 2 \end{cases} \quad \text{and} \quad gr(Q\Gamma_0(\mathbb{Z}_m)) = \begin{cases} 4, & \text{if } k = 2 \\ 3, & \text{if } k > 2 \end{cases}.$$

(2) *the domination number, the chromatic number and the clique number of $Q\Gamma_0(\mathbb{Z}_m)$ are k.*

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