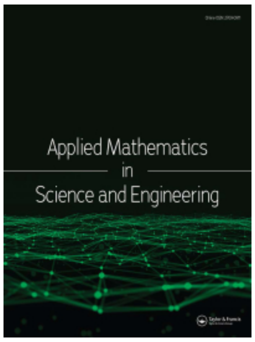

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Coefficients bounds for a subclass of *q*-bi-starlike functions associated with the generalized *q*-Lommel polynomials

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ABSTRACT
Orthogonal *q*-polynomials, both new and old, have witnessed a huge and revived attention in recent years, because of their applications in many diverse areas of mathematics and other sciences. In Geometric Function Theory, different subclasses of analytic and bi-univalent functions have been investigated and studied involving different orthogonal *q*-polynomials. In our present investigation, motivated by these recent research going on, first, we define some new subclasses of *q*-bi-starlike functions with the help of certain *q*-derivative operator which involving the generalized *q*-Lommel polynomials and *q*-Chebyshev polynomials. We then obtain the initial coefficients bounds for our defined function classes. Furthermore, the Fekete–Szegö inequalities are obtained for these defined function classes.

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1. Introduction and definitions

Researchers in Geometric Function Theory have used q-calculus to build and explore some unique subclasses of analytic and bi-univalent functions. The q-calculus was officially established when Jackson introduced q-integrals and q-derivatives (D_q) in 1909 [1, 2]. Furthermore, the q-calculus is applied in mechanics, statistics, number theory, relativity, combinatorics, control theory, and many other branches of sciences.

Quantum calculus, often known as q-calculus, in mathematics that expands classical calculus to non-commutative contexts. This idea is very important to understand in many

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areas of quantum study, especially when looking into q-deformed oscillators. In quantum mechanics, these oscillators are like traditional harmonic oscillators, but they work with a different set of algebraic rules. The area of q-calculus is used to analyse domains that are under study q-deformation, which involves relations with substituted q-analogues. This enables additional and flexible strategy for particular Quantum Field Theory (QFT). Moreover, they are utilized in Quantum Information Theory, a significant topic, particularly in the study of q-deformed quantum channels. q-Deformed Quantum Harmonic Oscillators have been used to define new models of quantum mechanics. For instance, they've been used to study particle behaviour in unusual potentials and non-trivial geometric contexts. They are involved in q-Deformed Quantum Optics by applying q-calculus in important research on variations of quantum optical systems that have been q-deformed. Quantum optics and information processing use q-deformed coherent and photon number states.

Mathematically q-deformed oscillators are defined by inserting a q as a parameter that modifies the exist relation of position and momentum $[\mathbf{x}, p] = i\hbar$ to $[\mathbf{x}, p] = i\hbar q^N$, where N is the number of oscillation. Similarly the q-deformed Quantum Fourier Transform is given by

$$\mid \mathbf{x}\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e_q^{\frac{2\Pi \mathbf{x} y}{N}} \mid y\rangle,$$

where e_q is the q-exponential function.

A special class of polynomials with applications in both q-calculus and q-quantum physics is the q-orthogonal polynomials. The polynomials in question are classical orthogonal polynomials, such as Legendre, Hermite, or Laguerre polynomials, with a q-deformation. For q-orthogonal polynomials, the orthogonality relation is modified to include the q-deformation parameter q. The deformation parameter q gives rise to unique properties of the q-orthogonal polynomials, such as changed recurrence relations and deformed orthogonality requirements. Furthermore, the q-orthogonal is essential to the study of combinatorics and statistical mechanics. Moreover, q-orthogonal polynomials frequently resemble q-special functions, such as q-binomials, q-exponential functions, and q-factorials. This field has slowly but steadily advanced. Aldweby et al. [3] constructed q-analogs of a number of operations using analytic functions by the use of convolution theory. In relation to analytical functions that comprise q-versions of hypergeometric functions, they examined the structure of q-operators. The q-calculus has garnered increasing attention from scholars, as evidenced by the numerous articles [4, 5] that include innovative findings and concepts.

Let $\mathcal{B}$ be the subclass of analytic functions λ in open unit disk.

$$\mathbb{M} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}, \tag{1}$$

which fulfil the condition for normalization given as

$$h(0) = h'(0) - 1 = 0.$$

The expression for Taylor Maclaurin series expansion of a function h is as follows:

$$h(z) = 1 + \sum_{i=1}^{\infty} a_i z^i \quad (z \in \mathbb{M}). \tag{2}$$

Let $\boldsymbol{S}$ be the class of functions in $\mathcal{B}$ that are univalent in $\mathbb{M}$. The analytic function contains starlike function which is denoted by $\mathbf{S}^*$, which contains set of normalized function $h \in \boldsymbol{S}$ which satisfies the given condition:

$$Re\left(\frac{zh'(z)}{h(z)}\right) > 0 \quad (\vee z \in h).$$

Let $h(z)$ and $l(z)$ be analytic functions in unit disk $\mathbb{M}$, $l(z)$ will be considered a subordinate of $h(z)$.

$$h(z) \prec l(z),$$

if there found a holomorphic mapping $w(z)$ in $\mathbb{M}$, called Schwarz function with $|w(z)| < 1$ and $w(0) = 0$, so the function can be expressed as

$$h(z) = l(w(z)).$$

Moreover, the function ψ will belong to the class $\mathcal{P}$ if it possesses the following representation:

$$\psi(z) = 1 + \sum_{i=1}^{\infty} a_i z^i$$

with

$$Re(\psi(z) > 0.$$

As known, a function $h \in \boldsymbol{S}$ has an inverse h^{-1} defined by

$$h^{-1}(h(z)) = z \quad (z \in \mathbb{M})$$

and

$$h^{-1}(h(w)) = w\left(|w| < r_0(h); r_0(h) \geq \frac{1}{4}\right),$$

where

$$h^{-1}(w) = l(w) = w - a_2 w^2 + \left(2a_2^2 - b_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots, \qquad (3)$$

if $h(z)$ and $h^{-1}(z)$ are univalent in $\mathbb{M}$, then function will be univalent.

Let Σ shows the class of bi-univalent function in the open unit disk $\mathbb{M}$ given by (2). Some examples of the class Σ of bi-univalent functions are given by

$$\frac{z}{1-z}, \log\frac{1}{1-z} \quad \text{and} \quad \log\sqrt{\frac{1+z}{1-z}}.$$

However, the well-known Koebe function is an example of class Σ given by

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots.$$

Lewin [6] shows $|a_2| < 1.51$ after studding bi-univalent functions class Σ. Following that, authors of [7] determine $|a_2| < \sqrt{2}$, while Netanyahu [8], on the other hand, showed that

$$\max_{h \in \Sigma} |a_2| = \frac{4}{3}.$$

The well-known subclasses of convex as well as starlike functions are $\mathbf{S}^*(\zeta)$ and $\mathcal{K}(\zeta)$ respectively having order $\zeta(0 \leq \zeta < 1)$, authors Brannan and Taha [9] defined $\mathbf{S}^*_\Sigma(\zeta)$ of bi-starlike functions and $\mathcal{K}_\Sigma(\zeta)$ of bi-convex functions of order $\zeta(0 \leq \zeta < 1)$, respectively, as subclasses of the bi-univalent function class Σ. They derived non-sharp limit on the first two Taylor–Maclaurin coefficients, $|a_2|$ and $|a_3|$, for each of the function classes $\mathbf{S}^*_\Sigma(\zeta)$ and $\mathcal{K}_\Sigma(\zeta)$.

Definition 1.1: If $q \in (0, 1)$, then q-number $[\beta]_q$ can be defined as

$$[\beta]_q := \begin{cases} \dfrac{1-q^\beta}{1-q} & (\beta \in \mathbb{C}), \\ \displaystyle\sum_{\beta=0}^{m-1} q^\beta = 1 = q + q^2 + \cdots + q^{m-1} & (\beta \in \mathbb{N}). \end{cases}$$

Definition 1.2: If $q \in (0, 1)$, then q-factorial $[\vartheta]_q!$ can be defined as

$$[\vartheta]_q := \begin{cases} 1 & (\vartheta = 0), \\ \Pi_{i=1}^m [i]_q & (m \in \mathbb{N}). \end{cases}$$

Definition 1.3 ([10, 11]): Let h be a function than q-derivative D_q of function h is given by

$$\left(D_q h\right)(z) = \frac{h(z) - h(qz)}{(1-q)z},$$

if we let $q \to 1-$, then we have

$$\lim_{q \to 1-} \{(D_q h)(z)\} = h'(z).$$

The operator of q-derivative D_q is an essential tool for studying the several subclasses of analytic functions given in Definition 4. Although the authors of [12] first introduce q-extension of starlike functions, while q-hypergeometric function was first used by Srivastava [13] and gave fundamental uses of it in geometric function theory. Many mathematicians have contributed extensively to the development of GFT [14–17]. Wongsaijai and Sukantamala extensively introduced and analysed new subclasses of starlike functions, as referenced in [18–20] expanded the idea of authors [18], introducing generalized subfamilies of q-starlike functions connected to Janowski functions. Moreover, the q-Mittag–Leffler functions have close-to-convexity properties which are explored by Srivastava and Bansal in their article [21]. Also, Mahmood et al. [22] studied conic region while Srivastava et al. [23] and Mahmood et al. [24] studied q-starlike function related to the Janowaski functions. It is also worth mentioning that, for the class of q-starlike functions, the third Hankel determinant upper bound was studied by Mahmood et al. [25]. In his recent study by Srivastava et al. [26], for a subclass of q-starlike functions, the bonds for the Hankel and Toeplitz determinants were given. To study on this subject, we may refer the readers to see [4, 5].

Definition 1.4 (see [12]): A function $h \in \boldsymbol{S}$ is said to be class $\mathbf{S}_q^*$ of q-starlike functions in $\mathbb{M}$ if

$$h'(0) - 1 = h(0) = 0 \tag{4}$$

and

$$\left| \frac{z(D_q h)(z)}{h(z)} - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}, \tag{5}$$

as $q \to 1-$, then close disk

$$\left| \omega - \frac{1}{1-q} \right| \leqq \frac{1}{1-q}$$

become right half of complex plane and $\mathbf{S}_q^*$in $\mathbb{M}$ become normalized $(\mathbf{S}^*)$ starlike function, by using principal of subordination between analytic functions [27]

$$\frac{z}{h(z)}(D_q h)(z) \prec h(z) \quad h(z) := \frac{1+z}{1-qz}.$$

The q-binomial expansion can be defined as follows.

Definition 1.5 ([28]): The expansion defined in the form of

$$(\mathbf{x} + a)_q^n = \sum_{j=0}^{n} q^{\frac{j(j-1)}{2}} \begin{bmatrix} n \\ j \end{bmatrix} a^j \mathbf{x}^{n-j} \tag{6}$$

is known as q-binomial expansion, where $\left[{n \atop j} \right]$ are q-binomial coefficient

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]_q!}{[j]_q![n-j]_q!}.$$

Al Salam and Ismail [28] used the q-Lommel polynomials [29] and gave a new q-polynomials as follows.

Definition 1.6 ([28]): The polynomial

$$K_n(\mathbf{x}, \alpha, \xi) = \sum_{j=0}^{\frac{n}{2}} \frac{(-\alpha; q).(q; q)_{n-j} \mathbf{x}^{n-2j} (-\xi)^j}{(-\alpha; q)(q; q)_j (q; q)_{n-j}} q^{j(j-1)} \tag{7}$$

is known as the generalized q-Lommel polynomials.

Theorem 1.1 ([28]): *The generalized q-Lommel polynomials satisfy*

$$k_{n+1}(\mathbf{x}; \alpha, \xi) = k_n(\mathbf{x}; \alpha, \xi)(1 + \alpha q^n)\mathbf{x} - k_{n-1}(\mathbf{x}; \alpha, \xi)\xi q^{n-1} \tag{8}$$

with initial condition

$$k_0(\mathbf{x}; \alpha, \xi) = 1 \quad \text{and} \quad k_1(\mathbf{x}; \alpha, \xi) = (1 + \alpha)\mathbf{x}.$$

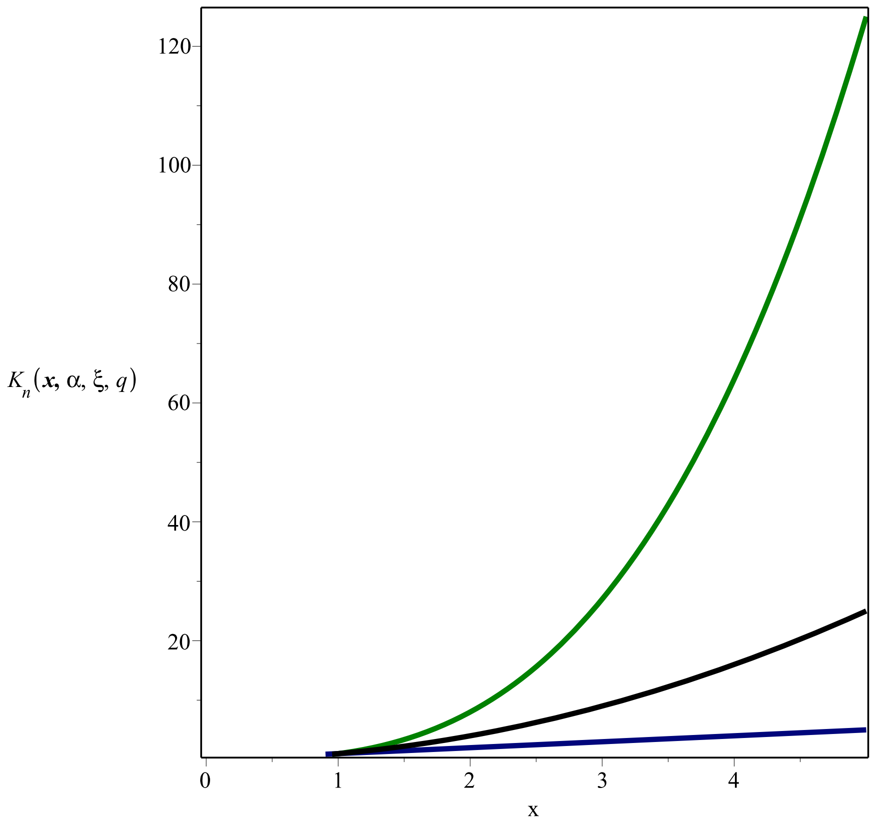

Figure 1. The above graph shows the graphical behaviour of $K_n(\mathbf{x}, \alpha, \xi)$ for the values of $q = 0.5, \alpha = -0.6, \xi = 0.2$ and $n = 1, 2, 3$ (blue, black and green) respectively.

Using the generalized q-Lommel polynomials, we establish the following.

Definition 1.7: Let $\psi(\mathbf{x}, z, \alpha, \xi, q)$ be defined as follows:

$$\psi(\mathbf{x}, z, \alpha, \xi, q) = 1 + \sum_{j=1}^{\infty} k_j(\mathbf{x}, z, \alpha, \xi, q) z^j. \tag{9}$$

Now a days, the usage of the q-orthogonal polynomials is significant in the context of Geometric Function Theory of complex analysis [30–32]. We define subclasses of analytic and bi-univalent functions by using the principle of subordination and q-starlike function as follows.

Definition 1.8: A function $h \in \mathbb{M}$ is said to be in the class $\widetilde{F}_{\Sigma}^{q,y}(t)$, if the following conditions are satisfied:

$$\left(\frac{z(D_q h)(z)}{h(z)}\right) \prec \psi(\mathbf{x}, z, \alpha, \xi, q) \quad \left(\frac{1}{2} < \mathbf{x} < 1, 0 < q < 1, z \in \mathbb{M}\right) \tag{10}$$

and

$$\left(\frac{z(D_q l)(w)}{l(w)}\right) \prec \psi(z,\mathbf{x},\alpha,\xi,q) \quad \left(\frac{1}{2} < \mathbf{x} < 1, 0 < q < 1, w \in \mathbb{M}\right). \tag{11}$$

We note from (9) that

$$\psi(z,\mathbf{x},\alpha,\xi,q) = 1 + k_1(z,\mathbf{x},\alpha,\xi,q)z + k_2(z,\mathbf{x},\alpha,\xi,q)z^2 + k_3(z,\mathbf{x},\alpha,\xi,q)z^3 + \cdots$$

while $z \in \mathbb{M}$ and $t \in (-1,1)$.

By using (8), we found the following relations:

$$\begin{aligned}
k_1(\mathbf{x};\alpha,\xi) &= \mathbf{x}(1+\alpha)\\
k_2(\mathbf{x};\alpha,\xi) &= \mathbf{x}^2(1+\alpha q)(1+\alpha) - \xi\\
k_3(\mathbf{x};\alpha,\xi) &= \mathbf{x}^3(1+\alpha q^2)(1+\alpha q)(1+\alpha) - \mathbf{x}(1+\alpha q^2)\xi - \mathbf{x}q(1+\alpha)\xi\\
k_4(\mathbf{x};\alpha,\xi) &= \mathbf{x}^4(1+\alpha q^3)(1+\alpha q^2)(1+\alpha q)(1+\alpha) - \mathbf{x}^2(1+\alpha q^2)\xi\\
&\quad - \mathbf{x}^2 q(1+\alpha q^3)(1+\alpha)\xi + \mathbf{x}^2 q^2(1+\alpha q)(1+\alpha)\xi + \xi^2 q^2.
\end{aligned}$$

To prove our main results, we need the following Lemma.

Lemma 1.1 (see [33]): *Consider the function m defined by*

$$m(z) = 1 + m_1 z + m_2 z^2 + \cdots$$

belong to the class $\boldsymbol{p}$ *of functions having a positive real part. Then*

$$|m_i| \leqq 2 \quad (i \in \mathbb{N}).$$

This last inequality is sharp.

2. Coefficients bounds for $h \in \widetilde{F}_{\Sigma}^{q,\mathbf{x}}(t)$

Theorem 2.1: *Let* $h \in \widetilde{F}_{\Sigma}^{q,\mathbf{x}}(t)$. *Then*

$$|a_2| \leq \frac{\mathbf{x}(1+\alpha)\sqrt{\mathbf{x}(1+\alpha)}}{\sqrt{\left|q^2\mathbf{x}(1+\alpha)[\mathbf{x}(1+\alpha) - \mathbf{x}(1+\alpha q) + 1] + \xi q^2\right|}}, \tag{12}$$

$$|a_3| \leq \frac{(\mathbf{x}(1+\alpha))^2}{q^2} + \frac{\mathbf{x}(1+\alpha)}{q^2} \tag{13}$$

and

$$|a_4| \leq \frac{5t^2(1+\alpha)^2}{2q^2(1+q)} + \frac{3(\mathbf{x}^2(1+\alpha q)(1+\alpha) - \xi) + 2\mathbf{x}(1+\alpha)}{q+q^2+q^3}. \tag{14}$$

Proof: Let $h \in \Sigma$ given by (2) be in the class $\widetilde{F}_{\Sigma}^{q,\mathbf{x}}(t)$. Then

$$\left(\frac{z(D_q h)(z)}{h(z)}\right) = \psi(\omega(z), \mathbf{x}; \alpha, \xi, q) \tag{15}$$

and

$$\left(\frac{z(D_q l)(z)}{l(z)}\right) = \psi(\varpi(w), \mathbf{x}; \alpha, \xi, q). \tag{16}$$

Let $m, n \in \boldsymbol{S}$ then

$$m(z) = \frac{1+\Lambda(z)}{1-\Lambda(z)} = 1 + m_1 z + m_2 z^2 + m_3 z^3 + \cdots$$

$$\Rightarrow \omega(z) = \frac{m(z)-1}{m(z)+1} \quad (z \in \mathbb{M}) \tag{17}$$

and

$$n(w) = \frac{1+\varepsilon(w)}{1-\varepsilon(w)} = 1 + n_1 w + n_2 w^2 + n_3 w^3 + \cdots$$

$$\Rightarrow \varepsilon(w) = \frac{n(w)-1}{n(w)+1} \quad (w \in \mathbb{M}). \tag{18}$$

It follows that from (17) and (18) that

$$\Lambda(z) = \frac{1}{2}\left[m_1 z + \left(m_2 - \frac{m_1^2}{2}\right) z^2 + \left(m_3 - m_1 m_2 + \frac{m_1^3}{4}\right) z^3 + \cdots\right] \tag{19}$$

and

$$\varepsilon(w) = \frac{1}{2}\left[n_1 w + \left(n_2 - \frac{n_1^2}{2}\right) w^2 + \left(n_3 - n_1 n_2 + \frac{n_1^3}{4}\right) w^3 + \cdots\right]. \tag{20}$$

From (19) and (20), applying $\psi(z, \mathbf{x}, \alpha, \xi, q)$ as given in (9), we see that

$$\begin{aligned}\psi(\Lambda(z), \mathbf{x}, \alpha, \xi, q) = 1 &+ \frac{k_1(\mathbf{x}, \alpha, \xi, q)}{2} m_1 z \\ &+ \left[\frac{k_1(\mathbf{x}, \alpha, \xi, q)}{2}\left(m_2 - \frac{m_1^2}{2}\right) + \frac{k_2(\mathbf{x}, \alpha, \xi, q)}{4} m_1^2\right] z^2 \\ &+ \frac{k_1(\mathbf{x}, \alpha, \xi, q)}{2}\left(m_3 - m_1 m_2 + \frac{m_1^3}{4}\right) \\ &+ \frac{k_2(\mathbf{x}, \alpha, \xi, q)}{2} m_1 \left(m_2 - \frac{m_1^2}{2}\right) + \frac{k_3(\mathbf{x}, \alpha, \xi, q)}{8} m_1^3 z^3 + \cdots,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\psi(\varepsilon(w), \mathbf{x}, \alpha, \xi, q) = 1 &+ \frac{k_1(\mathbf{x}, \alpha, \xi, q)}{2} n_1 w \\ &+ \left[\frac{k_1(\mathbf{x}, \alpha, \xi, q)}{2}\left(n_2 - \frac{n_1^2}{2}\right) + \frac{k_2(\mathbf{x}, \alpha, \xi, q)}{4} n_1^2\right] w^2\end{aligned}$$

$$+\left[\frac{k_1(\mathbf{x},\alpha,\xi,q)}{2}\cdot\left(m_3-m_1m_2+\frac{m_1^3}{4}\right)\right.$$
$$\left.+\frac{k_2(\mathbf{x},\alpha,\xi,q)}{2}n_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{k_3(\mathbf{x},\alpha,\xi,q)}{8}n_1^3\right]w^3+\cdots. \quad (22)$$

It follows from (15), (21), (16) and (22), we have

$$qa_2=\frac{k_1(\mathbf{x},\alpha,\xi,q)}{2}m_1, \quad (23)$$

$$(q+q^2)a_3-qa_2^2=\frac{k_1(\mathbf{x},\alpha,\xi,q)}{2}\left(m_2-\frac{m_1^2}{2}\right)+\frac{k_2(\mathbf{x},\alpha,\xi,q)}{4}m_1^2, \quad (24)$$

$$(q+q^2+q^3)a_4-(2q+q^2)a_2a_3+qa_2^3=\frac{k_1(\mathbf{x},\alpha,\xi,q)}{2}\cdot$$
$$\left(m_3-m_1m_2+\frac{m_1^3}{4}\right)+\frac{k_2(\mathbf{x},\alpha,\xi,q)}{2}m_1\left(m_2-\frac{m_1^2}{2}\right)$$
$$+\frac{k_3(\mathbf{x},\alpha,\xi,q)}{8}m_1^3, \quad (25)$$

$$-qa_2=\frac{k_1(\mathbf{x},\alpha,\xi,q)}{2}n_1, \quad (26)$$

$$q\{(1+2q)a_2^2-(1+q)a_3\}=\frac{k_1(\mathbf{x},\alpha,\xi,q)}{2}\left(m_2-\frac{m_1^2}{2}\right)+\frac{k_2(\mathbf{x},\alpha,\xi,q)}{4}m_1^2, \quad (27)$$

$$-q\{(2+3q+5q^2)a_2^3-(3+4q+5q^2)a_2a_3+(1+q+q^2)a_4\}$$
$$=\frac{k_1(\mathbf{x},\alpha,\xi,q)}{2}\cdot\left(n_3-n_1n_2+\frac{n_1^3}{4}\right)$$
$$+\frac{k_2(\mathbf{x},\alpha,\xi,q)}{2}n_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{k_3(\mathbf{x},\alpha,\xi,q)}{8}n_1^3. \quad (28)$$

Adding (23) and (26), we have

$$m_1=-n_1,\quad m_1^2=n_1^2\quad\text{and}\quad m_1^3=-n_1^3 \quad (29)$$

and

$$a_2^2=\frac{k_1^2(\mathbf{x},\alpha,\xi,q)(m_1^2+n_1^2)}{8q^2}. \quad (30)$$

Also, adding (24), (27) and applying (29) yields

$$2q^2a_2^2=\frac{k_1^3(\mathbf{x},\alpha,\xi,q)(m_2+n_2)}{2k_1^2(\mathbf{x},\alpha,\xi,q)-2(k_2(\mathbf{x},\alpha,\xi,q)-k_1(\mathbf{x},\alpha,\xi,q))}. \quad (31)$$

Applying (29) in (30) gives

$$n_1^2=\frac{4q^2a_2^2}{k_1^2(\mathbf{x},\alpha,\xi,q)}. \quad (32)$$

Putting (32) into (31) and with some calculations, we have

$$|a_2|^2 = \left| \frac{k_1^3(\mathbf{x}, \alpha, \xi, q)(m_2 + n_2)}{4q^2 k_1^2(\mathbf{x}, \alpha, \xi, q) - 4q^2(k_2(\mathbf{x}, \alpha, \xi, q) - k_1(\mathbf{x}, \alpha, \xi, q))} \right|.$$

Using Lemma 1.1 and triangular inequality, we have

$$|a_2| \leq \frac{\mathbf{x}(1+\alpha)\sqrt{\mathbf{x}(1+\alpha)}}{\sqrt{\left|q^2\mathbf{x}(1+\alpha)[\mathbf{x}(1+\alpha) - \mathbf{x}(1+\alpha q) + 1] + \xi q^2\right|}}. \tag{33}$$

Subtracting (27) from (24) and with some calculations, we have

$$a_3 = 2(q+q^2)\left[\frac{k_1^2(\mathbf{x}, \alpha, \xi, q)(m_1^2 + n_1^2)}{8q^2}\right] - \frac{k_1(\mathbf{x}, \alpha, \xi, q)}{2}(n_2 - m_2). \tag{34}$$

Using Lemma 1.1 and triangular inequality, we have

$$|a_3| \leq \frac{(\mathbf{x}(1+\alpha))^2}{q^2} + \frac{\mathbf{x}(1+\alpha)}{q^2}. \tag{35}$$

Subtracting (28) from (25), we have

$$\begin{aligned} a_4 = {} & k_1(\mathbf{x}, \alpha, \xi, q)\left(m_3 - n_3 - m_1(m_2 + n_2) + \frac{m_1^3}{2}\right) \\ & + \frac{k_2(\mathbf{x}, \alpha, \xi, q)}{2}(m_1(m_2 - n_2) - m_1^3) + \frac{k_3(\mathbf{x}, \alpha, \xi, q)}{4} m_1^3. \end{aligned} \tag{36}$$

Using Lemma 1.1 and triangular inequality, we have

$$|a_4| \leq \frac{5t^2(1+\alpha)^2}{2q^2(1+q)} + \frac{3(\mathbf{x}^2(1+\alpha q)(1+\alpha) - \xi) + 2\mathbf{x}(1+\alpha)}{q+q^2+q^3}, \tag{37}$$

which completes the proof. ■

3. Fekete–Szegö inequalities for the function class $\widetilde{F}_{\Sigma}^{q,\mathbf{x}}(t)$

In class $\boldsymbol{S}$, a function's *nth* coefficient is constrained by n, and the coefficient bounds provide insights into geometric properties of the function. The well-known problem of Fekete–Szegö [34] is to determine the greatest possible value for the coefficient functional $\Omega_{\sigma}(h) := |a_3 - \sigma a_2^2|$ through class $\boldsymbol{S}$ where $\sigma \in [0, 1]$, solved using the Loewner approach. This section focuses on establishing the upper limits of the coefficient functional $|a_3 - \delta a_2^2|$ for class $\widetilde{F}_{\Sigma}^{q,\mathbf{x}}(t)$.

Theorem 3.1: *Let $h \in \widetilde{F}_{\Sigma}^{q,\mathbf{x}}(t)$. Then for some $\delta \in \mathbb{R}$,*

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \dfrac{2\mathbf{x}(1+r)}{q(1+q)} & |\delta - 1| \geq \mathbf{\Phi}_1(\mathbf{x}, q, \alpha, \xi), \\ 2\varsigma(\mathbf{x}, q, \alpha, \xi)\,|1-\delta| & |\delta - 1| \leq \mathbf{\Phi}_1(\mathbf{x}, q, \alpha, s), \end{cases}$$

where

$$\mathbf{\Phi}_1(\mathbf{x}, q, \alpha, \xi) = \left[\frac{\mathbf{x}\alpha(1-q)}{(1+q)(1+\alpha)} + \frac{q}{\mathbf{x}(1+q)(1+\alpha)} + \frac{\xi}{\mathbf{x}(1+q)(1+\alpha)^2}\right]$$

and

$$\varsigma(\mathbf{x}, q, \alpha, \xi) = \frac{\mathbf{x}^3(1+\alpha)^3}{q^2[\mathbf{x}^2(1+\alpha)^2 - \mathbf{x}^2(1+\alpha q)(1+\alpha) + \xi - \mathbf{x}(1+\alpha)]}$$

Proof: From (31) and (34), we have

$$\begin{aligned} a_3 - \delta a_2^2 &= \frac{(1-\delta)k_1^3(\mathbf{x};\alpha,\xi)(m_2+n_2)}{4q^2\left[k_1^2(\mathbf{x};\alpha,\xi) - k_2(\mathbf{x};\alpha,\xi) + k_1(\mathbf{x};\alpha,\xi)\right]} \\ &\quad + \frac{k_1(\mathbf{x};\alpha,\xi)[m_2 - n_2]}{4(q+q^2)} \\ &= \frac{k_1(\mathbf{x};\alpha,\xi)}{4}\left[\left(\tau(\delta) + \frac{1}{q+q^2}\right)m_2 + \left(\tau(\delta) - \frac{1}{q+q^2}\right)n_2\right], \end{aligned}$$

where

$$\tau(\delta) = \frac{(1-\delta)k_1^2(\mathbf{x};\alpha,\xi)}{q^2\left[k_1^2(\mathbf{x};\alpha,\xi) - k_2(\mathbf{x};\alpha,\xi) + k_1(\mathbf{x};\alpha,\xi)\right]}.$$

Applying Lemma 1.1, we have

$$\left|a_3 - \delta a_2^2\right| = k_1\left[\tau(\delta) + \frac{1}{q+q^2}\right]. \qquad \blacksquare$$

4. Coefficients bounds and Fekete–Szegö inequalities for $\widetilde{\mathbf{A}}_{\Sigma}^{q,\mathbf{x}}(t)$

The objective of this section is to make inquire about q-Chebyshev polynomial to derive initial coefficient estimates for a subclass of q-Starlike and bi-q-Starlike functions involving the q-Chebyshev polynomials of the second kind. In addition, the Fekete–Szegö inequalities for the class $\widetilde{\mathbf{A}}_{\Sigma}^{q,\mathbf{x}}(t)$ are established.

Theorem 4.1 ([35]): *The q-Chebyshev polynomials of the second kind satisfy*

$$I_m(t,\mathbf{x},q) = (1+q^m)tI_{m-1}(t,\mathbf{x},q) + q^{m-1}yI_{m-2}(t,\mathbf{x},q) \tag{38}$$

with initial values

$$I_0(t,\mathbf{x},q) = 1 \quad \text{and} \quad I_1(t,\mathbf{x},q) = t(1+q).$$

Remark 4.1: It is clear that

$$I_m(t) = I_m(t, -1, 1).$$

The term $I_m(t)$ represents the conventional Chebyshev polynomial of the second kind.

Using q-Chebyshev polynomials, we can define the following subclasses of analytic and bi-univalent functions.

Definition 4.1: Let $\varphi(t, \mathbf{x}, q)$ be defined as follows:

$$\varphi(t, \mathbf{x}, q) = 1 + \sum_{p=1}^{\infty} I_p(t, \mathbf{x}, q) z^p. \tag{39}$$

We define subclasses of analytic and bi-univalent function by using principle of subordination and q-starlike function.

Definition 4.2: A function $h \in \mathbb{M}$ is said to be in the class $\widetilde{\mathbf{A}}_{\Sigma}^{q,\mathbf{x}}(t)$, if the following conditions are satisfied:

$$\left(\frac{z(D_q h)(z)}{h(z)}\right) \prec \varphi(t, \mathbf{x}, q) \quad \left(\frac{1}{2} < \mathbf{x} < 1, 0 < q < 1, z \in \mathbb{M}\right) \tag{40}$$

and

$$\left(\frac{z(D_q l)(w)}{l(w)}\right) \prec \varphi(t, \mathbf{x}, q) \quad \left(\frac{1}{2} < \mathbf{x} < 1, 0 < q < 1, w \in \mathbb{M}\right). \tag{41}$$

We note from (9) that

$$\varphi(z, t, \mathbf{x}, q) = 1 + I_1(t, \mathbf{x}, q) z + I_2(t, \mathbf{x}, q) z^2 + I_3(t, \mathbf{x}, q) z^3 + \cdots$$

while $z \in \mathbb{M}$ and $t \in (-1, 1)$.

From (38), we derive the following relations:

$$\begin{aligned}
I_1(t, \mathbf{x}, q) &= t(1+q) \\
I_2(t, \mathbf{x}, q) &= t^2(1+q)(1+q^2) + q\mathbf{x} \\
I_3(t, \mathbf{x}, q) &= t^3(1+q)(1+q^2)(1+q^3) + qt(1+q)(1+q^2)\mathbf{x} \\
I_4(t, \mathbf{x}, q) &= (1+q)(1+q^2)(1+q^3)(1+q^4)t^4 \\
&\quad + q(1+q)(1+q^2)(1+q^4+q^2)y^2 t + q^4\mathbf{x}.
\end{aligned}$$

Theorem 4.2: *Let* $h \in \widetilde{\mathbf{A}}_{\Sigma}^{q,\mathbf{x}}(t)$. *Then*

$$|a_2| \leq \frac{(1+q)t\sqrt{(1+q)t}}{\sqrt{\left|q^2(1+q)^2 t^2 - [t^2(1+q)(1+q^2) + q\mathbf{x} - t(1+q)]q^2\right|}}, \tag{42}$$

$$|a_3| \leq \frac{(1+q)^2 t^2}{q^2} + \frac{(1+q)t}{q+q^2} \tag{43}$$

and

$$|a_4| \leq \frac{5(1+q)}{2q^2} t^2 + \frac{3((1+q^2)(1+q)t^2 + qy) + 2(1+q)t}{q+q^2+q^3}$$

Proof: The proof of Theorem 4.2 is similar to that of the proof of Theorem 2.1, we here chose to omit the analogue details of the proof of 4.2. ■

Theorem 4.3: *Let* $h \in \widetilde{\mathbf{A}}_{\Sigma}^{q,\mathbf{x}}(t)$. *Then for some* $\delta \in \mathbb{R}$

$$\left|a_3 - \delta a_2^2\right| \leq \begin{cases} \dfrac{2t}{q} & |\delta - 1| \geq \lambda(t, \mathbf{x}, q), \\ 2\varrho(\mathbf{x}, q, \alpha, s)\,|1-\delta| & |\delta - 1| \leq \lambda(t, \mathbf{x}, q), \end{cases}$$

where

$$\lambda(t, \mathbf{x}, q) = \left[\frac{q^2(1-q)}{(1+q)} + \frac{q}{t(1+q)} - \frac{q^2\mathbf{x}}{t^2(1+q)^2}\right]$$

and

$$\varrho(t, \mathbf{x}, q) = \frac{t^3(1+q)^3}{q^2[t^2(1+q)^2 - t^2(1+q)(1+q^2) + q\mathbf{x} + t(1+q)]}.$$

Proof: The proof is simple, therefore omitted. ■

5. Conclusion

Orthogonal q-polynomials, both new and old, have witnessed a huge and revived attention in recent years, because of their applications in many diverse areas of Mathematics and other Sciences, as mentioned in the introductory section. In Geometric Function Theory, different subclasses of analytic and bi-univalent functions have been investigated and studied involving different Orthogonal q-polynomials. Here in our present investigation, we have been motivated by those recent research going on and have defined some new subclasses of q-bi-starlike functions with the help of certain q-derivative operator which involving the generalized q-Lommel polynomials and q-Chebyshev polynomials. We have then obtained the initial coefficients bounds for our defined functions classes. Furthermore, the Fekete–Szegö inequalities have been obtained for these defined functions classes of q-starlike and q-bi-starlike functions, related to q-Lommel polynomials and q-Chebyshev polynomials.

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The authors declare that they have not used artificial intelligence tools in the creation of this article.

References

[1] Jackson FH. On q-functions and a certain difference operator. Earth Environ Sci Trans R Soc Edinb. 1909;46:253–281. doi: 10.1017/S0080456800002751

[2] Jackson FH. On q-definiteintegrals. Q J Pure Appl Math. 1910;41:193–203.

[3] Aldweby H, Darus M. On harmonic meromorphic functions associated with basic hypergeometric functions. Sci World J. 2013;2013:164287. doi: 10.1155/tswj.v2013.1

[4] Khan WA, Araci S, Acikgoz M, et al. Laguerre-based Hermite–Bernoulli polynomials associated with bilateral series. Tbil Math J. 2018;11:111–12.

[5] Duran U, Acikgoz M, Esi A, et al. A note on the (p,q)-Hermite polynomials. Appl Math Inf Sci. 2018;12:227–231. doi: 10.18576/amis/120122ISRN Math. Anal. 2013, 2013, 382312.

[6] Lewin M. On a coefficient problem for bi-univalent functions. Proc Am Math Soc. 1967;18(1):63–68. doi: 10.1090/proc/1967-018-01

[7] Brannan DA, Clunie JG. Aspect of Contemporary Complex Analysis. Proceedings of the NATO Advanced study Institute held at the University of Durham, Durham; July 1–20, 1979. Academic Press, New York and London, 1980.

[8] Netanyahu E. The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. Arch Rational Mech Anal. 1969;32:100–112. doi: 10.1007/BF00247676

[9] Brannan DA, Taha T. On some classes of bi-univalent functions. Babes-Bolyai Math. 1986;31:70–77.

[10] Jackson FH. On q-definite integrals. Quart J Pure Appl Math. 1910;41:193–203.

[11] Jackson FH. q-Difference equations. Amer J Math. 1910;32:305–314. doi: 10.2307/2370183

[12] Ismail ME-H, Merkes E, Styer D. A generalization of starlike functions. Complex Variables Theory Appl. 1990;14:77–84.

[13] Srivastava HM. Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in *Univalent Functions, Fractional Calculus, and Their Applications* (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329–354, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1989.

[14] Aldweby H, Darus M. Some subordination results on q-analogue of Ruscheweyh differential operator. Abstr Appl Anal. 2014;2014:1–6. doi: 10.1155/2014/958563Article ID 958563.

[15] Ezeafulukwe UA, Darus M. Certain properties of q-hypergeometric functions. Int J Math Math Sci. 2015;2015:1–9. doi: 10.1155/2015/489218Article ID 489218.

[16] Mahmood S, Raza N, Abujarad ESA, et al. Geometric properties of certain classes of analytic functions associated with a q-integral operator. Symmetry. 2019;11:1–14. Article ID 719.

[17] Srivastava HM, Khan B, Khan N, et al. A generalized conic domain and its applications to certain subclasses of analytic functions. Rocky MT J Math. 2019;49:2325–2346.

[18] Wongsaijai B, Sukantamala N. Certain properties of some families of generalized starlike functions with respect to q-calculus. Abstr Appl Anal. 2016;2016:1–8. doi: 10.1155/2016/6180140 Article ID 6180140.

[19] Srivastava HM, Tahir M, Khan B, et al. Some general classes of q-starlike functions associated with the Janowski functions. Symmetry. 2019;11:1–14. Article ID 292.

[20] Srivastava HM, Tahir M, Khan B, et al. Some general families of q-starlike functions associated with the Janowski functions. Filomat. 2019;33:2613–2626. doi: 10.2298/FIL1909613S

[21] Srivastava HM, Bansal D. Close-to-convexity of a certain family of q-Mittag–Leffler functions. J Nonlinear Var Anal. 2017;1:61–69.

[22] Mahmood S, Jabeen M, Malik SN, et al. Some coefficient inequalities of q-Starlike functions associated with conic domain defined by q-derivative. J Funct Spaces. 2018;2018:1–13. Article ID 8492072.

[23] Srivastava HM, Khan B, Khan N, et al. Coefficient inequalities for q-starlike functions associated with the Janowski functions. Hokkaido Math J. 2019;48:407–425. doi: 10.14492/hokmj/1562810517

[24] Mahmood S, Ahmad QZ, Srivastava HM, et al. A certain subclass of meromorphically q-starlike functions associated with the Janowski functions. J Inequal Appl. 2019;2019:1–11. doi: 10.1186/s13660-019-1955-4Article ID 88.

[25] Mahmood S, Srivastava HM, Khan N, et al. Upper bound of the third Hankel determinant for a subclass of q-starlike functions. Symmetry. 2019;11:1–13. Article ID 347.

[26] Srivastava HM, Ahmad QZ, Khan N, et al. Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain. Mathematics. 2019;7:1–15. Article ID 181.

[27] Ö. Uçar HE. Coefficient inequality for q-starlike functions. Appl Math Comput. 2016;276: 122–126.

[28] Al-Salam W, Ismail M. Orthogonal polynomials associated with the Rogers–Ramanujan continued fraction. Pacific Journal of Mathematics. 1983;104:269–283. doi: 10.2140/pjm

[29] Ismail MEH. The zeros of basic Bessel functions, the functions $J_{p+ax}(x)$ and associated orthogonal polynomials. J Math Anal Appl. 1982;86:1–19. doi: 10.1016/0022-247X(82)90248-7

[30] Hu Q-X, Shaba TG, Younis J, et al. Applications of q-derivative operator to subclasses of bi-univalent functions involving Gegenbauer polynomial. Appl Math Sci Engrg. 2022;30:501–520. doi: 10.1080/27690911.2022.2088743

[31] Zhang C, Khan B, Shaba TG, et al. Applications of q-Hermite polynomials to subclasses of analytic and bi-univalent functions. Fractal Fract. 2022;6:1–15. Article ID 420.

[32] Khan B, Liu Z-G, Shaba TG, et al. Applications of-derivative operator to the subclass of bi-univalent functions involving q-Chebyshev polynomials. J Math. 2022;2022:7. Article ID 8162182.

[33] Duren PL. Univalent functions. New York, Berlin, Heidelberg and Tokyo: Springer-Verlag; 1983. (Grundlehren der Mathematischen Wissenschaften, Band; 259).

[34] Fekete M, Szego G. Eine bemerkung uber ungerade schlichte funktionen. J Lond Math Soc. 1933;s1-8:85–89. doi: 10.1112/jlms/s1-8.2.85

[35] Cigler J. A simple approach to q-Chebyshev polynomial, arXiv preprint arXiv:1201.4703, 2012.