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Applications of Symmetric Conic Domains to a Subclass of q -Starlike Functions

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Abstract: In this paper, the theory of symmetric q -calculus and conic regions are used to define a new subclass of q -starlike functions involving a certain conic domain. By means of this newly defined domain, a new subclass of normalized analytic functions in the open unit disk E is given. Certain properties of this subclass, such as its structural formula, necessary and sufficient conditions, coefficient estimates, Fekete–Szegő problem, distortion inequalities, closure theorem and subordination results, are investigated. Some new and known consequences of our main results as corollaries are also highlighted.

Keywords: quantum (or q -) calculus; symmetric quantum (or q -) calculus; symmetric q -derivative; q -starlike functions; symmetric conic domains

MSC: Primary 05A30, 30C45; Secondary 11B65, 47B38



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1. Introduction and Definitions

Let \mathcal{A} denote the class of all analytic functions f which are analytic in the open unit disk

$$E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and series expansion is

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1)$$

The usage of normalize functions in geometric function theory is significant. They are used to map everything inside a unit circle, where everything is convergent. Using this concept, many different subclasses of analytic functions have been defined and studied.

Furthermore, let $\mathcal{S} \subset \mathcal{A}$ represents the set of all univalent in E (see [1,2]). The classes of uniformly convex (UCV) and uniformly starlike (UST) functions were introduced by Goodman [3] and are defined analytically as:

$$f \in UCV \iff f \in \mathcal{A} \text{ and } \left| \frac{zf''(z)}{f'(z)} \right| < \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\}, \quad z \in E$$

and

$$f \in UST \iff f \in \mathcal{A} \text{ and } \left| \frac{zf'(z)}{f(z)} - 1 \right| < \Re \left\{ \frac{zf'(z)}{f(z)} \right\}, \quad z \in E.$$

The classes of k -uniformly convex (k - \mathcal{UCV}) and k -uniformly starlike (k - \mathcal{UST}) functions were introduced by Kanas and Wisniowska in [4] and are defined analytically as:

$$f(z) \in k\text{-}\mathcal{UST} \iff f(z) \in \mathcal{A} \text{ and } 1 > k \left| \frac{zf'(z)}{f(z)} - 1 \right| - \Re \left\{ \frac{zf'(z)}{f(z)} \right\}, \quad z \in E,$$

and

$$f(z) \in k\text{-}\mathcal{UCV} \iff f(z) \in \mathcal{A} \text{ and } 1 > k \left| \frac{zf''(z)}{f'(z)} \right| - \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\}, \quad z \in E.$$

Note that

$$f(z) \in k\text{-}\mathcal{UCV} \iff zf'(z) \in k\text{-}\mathcal{ST}.$$

The above-defined function classes were studied and investigated by a number of well-known mathematicians; see for details [1–6].

Two analytic functions f and g are subordinate to each other, (written as $f(z) \prec g(z)$), if there exists a Schwarz function $s(z)$, which is analytic in E with

$$s(0) = 0 \quad \text{and} \quad |s(z)| < 1$$

such that

$$f(z) = g(s(z)).$$

Additionally, if $g(z) \in \mathcal{S}$ in E , then we have (see [1,2])

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(E) \subset g(E).$$

The convolution of two analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (z \in E)$$

is defined as:

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

Let \mathcal{P} represent the set of all of Carathéodory functions and every analytic function $p \in \mathcal{P}$, having series of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and satisfying the condition

$$\Re(p(z)) > 0, \quad z \in E.$$

In terms of these analytic functions, many different subclasses of starlike and convex functions have been defined already.

We have discussed above that Kanas and Wisniowska [4] introduced the class of k -uniformly convex functions (k - \mathcal{UCV}) and the corresponding class of k -starlike functions (k - \mathcal{ST}) and then defined these classes subject to the conic domain Ω_k , ($k \geq 0$) as follows:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}, \tag{2}$$

or

$$\Omega_k = \{w : \Re w > k|w - 1|\}.$$

For $k = 0$, then (2) becomes the right half plane; for $0 < k < 1$, it becomes a hyperbola, for $k = 1$ it becomes a parabola and for $k > 1$, it becomes an ellipse. The generalization

of theses discussed by Shams et al. in [7] is subject to the conic domain $\Omega_{k,\gamma}$ ($k \geq 0$), $\gamma \in \mathbb{C} \setminus \{0\}$, which is

$$\Omega_{k,\gamma} = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2 + \gamma} \right\},$$

or

$$\Omega_{k,\gamma} = \{w : \Re w > k|w-1| + \gamma\}.$$

For this conic domain, the function $\widetilde{p_{k,\gamma}}(z)$ plays the role of the extremal function for the conic domain $\Omega_{k,\gamma}$.

$$\widetilde{p_{k,\gamma}}(z) = \begin{cases} \frac{1+z}{1-z} & \text{for } k = 0, \\ 1 + \frac{2\gamma}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & \text{for } k = 1, \\ 1 + \frac{2\gamma}{1-k^2} H_1(k, z) & \text{for } 0 < k < 1, \\ 1 + \frac{\gamma}{k^2-1} H_1(k, z) + \frac{\gamma}{1-k^2} & \text{for } k > 1, \end{cases} \tag{3}$$

where

$$H_1(k, z) = \sinh^2 \left\{ \left(\frac{2}{\pi} \arccos k \right) \arctan h\sqrt{z} \right\}$$

$$H_2(k, z) = \sin \left(\frac{\pi}{2K(i)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(ix)^2}} dx \right)$$

and $i \in (0, 1)$, $k = \cosh \left(\frac{\pi K'(i)}{4K(i)} \right)$, $K(i)$ is the first kind of Legendre’s complete elliptic integral. For details (see [4]). Indeed, from (3), we have

$$\widetilde{p_{k,\alpha}}(z) = 1 + Q_1z + Q_2z^2 + \dots, \tag{4}$$

where

$$Q_1 = \begin{cases} \frac{2\gamma \left(\frac{2}{\pi} \arccos k \right)^2}{1-k^2} & \text{for } 0 \leq k < 1, \\ \frac{8\gamma}{\pi^2} & \text{for } k = 1, \\ H_3(k, z) & \text{for } k > 1, \end{cases} \tag{5}$$

$$Q_2 = \begin{cases} \frac{\left(\frac{2}{\pi} \arccos k \right)^2 + 2}{3} Q_1 & \text{for } 0 \leq k < 1, \\ \frac{2}{3} Q_1 & \text{for } k = 1, \\ H_4(k, z) Q_1 & \text{for } k > 1, \end{cases} \tag{6}$$

and

$$H_3(k, z) = \frac{\pi^2 \gamma}{4(1+t)\sqrt{t}K^2(t)(k^2-1)},$$

$$H_4(k, z) = \frac{4K^2(t)(t^2+6t+1) - \pi^2}{24K^2(t)(1+t)\sqrt{t}}.$$

In the twentieth century the researchers have developed a great interest in the study of theory of q -calculus and its numerous applications in the fields of mathematics and physics. In defining the q -analogous of the derivative and integral operator and providing few of their applications, Jackson [8] was one of the pioneer researchers. It has several applications in number theory, combinatorics, orthogonal polynomials, fundamental hyper-geometric functions, and other fields of mathematics, including quantum mechanics, mechanics

and relativity theory. Furthermore, quantum calculus has been proven to be a branch of the more comprehensive mathematical area of time scale calculus. For both discrete and continuous domains, time scales provide a coherent framework for studying dynamic equations. Many researchers provided useful applications for q -analysis in domains of mathematics; see [9–18]. Currently, operators of basic (or q -) calculus and fractional q -calculus and their applications have been elaborated by Srivastava in [19]. Considering the importance of q -operator calculus theory, researchers have intensively explored their applications in various fields; see [20–25].

In many areas of mathematics, the symmetric properties of functions play an important role in problem solving. Symmetry’s importance has been demonstrated in a variety of fields, including biology, chemistry, and psychology. Recently, in [26], Zhang et al. studied the application of a q -symmetric difference operator in harmonic univalent functions. Additionally, Khan et al. [27] defined the symmetric conic domain by using symmetric q -calculus operator theory and investigated q -starlike functions in this domain. Very recently, Khan et al. [28] defined and explored a new subclass of analytic and bi-univalent function involving certain q -Chebyshev polynomials by means of a symmetric q -operator. The symmetric q -calculus finds its applications in different fields, especially in quantum mechanics; see for example [29,30].

Here we present a few basic definitions, as well as a concept of q -calculus and symmetric q -calculus which will help us in further studies.

Definition 1 ([31]). *The q -number $[t]_q$ for $q \in (0, 1)$ is defined as:*

$$[t]_q = \begin{cases} \frac{1 - q^t}{1 - q}, & (t \in \mathbb{C}). \end{cases}$$

For $(t = n \in \mathbb{N})$, then

$$[t]_q = \sum_{k=0}^{n-1} q^k$$

and the q -gamma function is defined by:

$$\frac{\Gamma_q(t + 1)}{\Gamma_q(t)} = [t]_q$$

and

$$\Gamma_q(1) = 1.$$

Definition 2 ([27]). *For $n \in \mathbb{N}$, the symmetric q -number is defined as:*

$$[\widetilde{n}]_q = \frac{q^{-n} - q^n}{q^{-1} - q}, \quad [\widetilde{0}]_q = 0.$$

Remark 1. *The symmetric q -number can not reduce to a q -number.*

Definition 3 ([27]). *The q -factorial $[n]_q!$ for $q \in (0, 1)$ is defined as:*

$$[n]_q! = \prod_{k=1}^n [k]_q, \quad (n \in \mathbb{N}) \tag{7}$$

and

$$[0]_q! = 1.$$

Definition 4 ([27]). For any $n \in \mathbb{Z}^+ \cup \{0\}$, the symmetric q -number shift factorial is defined as:

$$[n]_q! = \begin{cases} [n]_q [n-1]_q [n-2]_q \dots [2]_q [1]_q, & n \geq 1 \\ 1 & n = 0. \end{cases}$$

Note that

$$\lim_{q \rightarrow 1^-} [n]_q! = n!.$$

Definition 5 ([32]). Let $f \in \mathcal{A}$; then, the symmetric q -derivative operator is defined by:

$$(\widetilde{D})_q f(z) = \frac{1}{z} \left(\frac{f(qz) - f(q^{-1}z)}{q - q^{-1}} \right), \quad z \in E. \tag{8}$$

We can write (8) as:

$$(\widetilde{D})_q f(z) = 1 + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}.$$

Note that

$$(\widetilde{D})_q z^n = [n]_q z^{n-1}, \quad (\widetilde{D})_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1},$$

and

$$\lim_{q \rightarrow 1^-} (\widetilde{D})_q f(z) = f'(z).$$

Definition 6 ([27]). Let $f \in \mathcal{A}$ and $0 < q < 1$; then $f \in \widetilde{S}_q^*$ if

$$f(0) = f'(0) = 1$$

and

$$\left| \frac{z(\widetilde{D})_q f(z)}{f(z)} - \frac{1}{1 - \frac{q}{q^{-1}}} \right| \leq \frac{1}{1 - \frac{q}{q^{-1}}}. \tag{9}$$

Using the definition of subordination, we can write the condition in (9) as:

$$\frac{w(\widetilde{D})_q g(z)}{g(z)} \prec \frac{1+z}{1 - \frac{q}{q^{-1}}z}.$$

Recently, several classes of analytic and univalent functions in different types of domains have investigated in [21,22,27,33–43]. For example, let $p(z)$ be an analytic in E and $p(0) = 1$. Then:

- (i) The image domain of E under $p(z)$ lies in right half plane if $p(z) \prec \frac{1+z}{1-z}$ (see [3]);
- (ii) For $-1 \leq B < A \leq 1$, the image of E under $p(z)$ lies inside a circle centered on real axis if $p(z) \prec \frac{1+Az}{1+Bz}$ (see [44]);
- (iii) For $0 \leq \alpha < 1$, in [4,34], Kanas showed that the image of E under $p(z)$ lies inside the conic domain Ω_k and $\Omega_{k,\alpha}$ if

$$p(z) \prec p_{k,\alpha}(z);$$

- (iv) For $\gamma \in \mathbb{C} \setminus \{0\}$, in [7], Shams et al. showed that the image of U under $p(z)$ lies inside the conic domain $\Omega_{k,\gamma}$ if

$$p(z) \prec p_{k,\gamma}(z).$$

By taking motivation from the above-cited works, we define the following domain:

Definition 7. Let $k \in [0, \infty)$, $q \in (0, 1)$ and $\gamma \in \mathbb{C} \setminus \{0\}$. A function $p(z)$ is said to be in the class $k\text{-}\widetilde{\mathcal{P}}_{q,q^{-1},\gamma}(z)$ if and only if

$$p(z) \prec \widetilde{p_{k,q,q^{-1},\gamma}}(z) \tag{10}$$

where

$$\widetilde{p_{k,q,q^{-1},\gamma}} = \frac{2q^{-1}\widetilde{p_{k,\gamma}}(z)}{(q^{-1} + q) + (q^{-1} - q)\widetilde{p_{k,\gamma}}(z)} \tag{11}$$

and $\widetilde{p_{k,\gamma}}(z)$ is given by (3). Geometrically, the function $p(z) \in k\text{-}\widetilde{\mathcal{P}}_{q,q^{-1},\gamma}(z)$ takes on all values from the domain $\widetilde{\Omega_{k,q,q^{-1},\gamma}}$ which is defined as:

$$\widetilde{\Omega_{k,q,q^{-1},\gamma}} = \gamma\widetilde{\Omega_{k,q,q^{-1}}} + (1 - \gamma), \tag{12}$$

where

$$\widetilde{\Omega_{k,q,q^{-1}}} = \left\{ w : \Re \left(\frac{(q^{-1} + q)w(z)}{(q - q^{-1})w(z) + 2q^{-1}} \right) > k \left| \frac{(q^{-1} + q)w(z)}{(q - q^{-1})w(z) + 2q^{-1}} - 1 \right| \right\}.$$

Remark 2. (i) First of all, we see that

$$\lim_{q \rightarrow 1^-} \widetilde{\Omega_{k,q,q^{-1},\gamma}} = \Omega_{k,\gamma} \quad (\gamma \in \mathbb{C} \setminus \{0\})$$

where $\Omega_{k,\gamma}$ is the conic domain considered by Shams et al. [7]. Secondly, we have

$$\lim_{q \rightarrow 1^-} \widetilde{\Omega_{k,q,q^{-1},1}} = \Omega_k,$$

where Ω_k is the conic domain considered by Kanas and Wisniowska [34]. Thirdly, we have

$$\lim_{q \rightarrow 1^-} k\text{-}\mathcal{P}_{q,q^{-1},1} = \mathcal{P}(p_k),$$

where $\mathcal{P}(p_k)$ is the well-known class introduced by Kanas and Wisniowska [34]. Lastly, we have

$$\lim_{q \rightarrow 1^-} 0\text{-}\mathcal{P}_{q,q^{-1},1} = \mathcal{P},$$

where \mathcal{P} is the well-known class of analytic functions with a positive real part.

Using the above-defined domain, we now define the following subclass of certain analytic functions:

Definition 8. An analytic function $f \in k\text{-}\mathcal{UST}(q, q^{-1}, \gamma)$ if it satisfies the condition

$$\Re \left\{ 1 + \frac{1}{\gamma} \left(\mathcal{J}(q, q^{-1}, f(z)) - 1 \right) \right\} > k \left| \frac{1}{\gamma} \left(\mathcal{J}(q, q^{-1}, f(z)) - 1 \right) \right|, \tag{13}$$

or equivalently

$$\mathcal{J}(q, q^{-1}, f(z)) \in k\text{-}\widetilde{\mathcal{P}}_{q,q^{-1},\gamma}, \tag{14}$$

where

$$\mathcal{J}(q, q^{-1}, f(z)) = \frac{(q^{-1} + q) \frac{z(\overline{D})_q f(z)}{f(z)}}{(q - q^{-1}) \frac{z(\overline{D})_q f(z)}{f(z)} + 2q^{-1}}. \tag{15}$$

Remark 3. First of all, we have (see [34])

$$\lim_{q \rightarrow 1^-} k\text{-UST}(q, q^{-1}, \gamma) = \mathcal{S}_{k,\gamma}^*$$

Secondly, it could be seen that (see [4])

$$\lim_{q \rightarrow 1^-} k\text{-UST}(q, q^{-1}, 1) = k\text{-UCV}.$$

Geometrically, the function $f \in \mathcal{A}$ is in the class $k\text{-UST}(q, q^{-1}, \gamma)$, if and only if the function $\mathcal{J}(q, q^{-1}, f(z))$ takes all values in the conic domain $\widetilde{\Omega}_{k,q,q^{-1},\gamma}$ given by (12). Taking this geometrical interpretation into consideration, one can rephrase the above definition as:

Definition 9. An analytic function $f \in k\text{-UST}(q, q^{-1}, \gamma)$ if and only if

$$\mathcal{J}(q, q^{-1}, f(z)) \prec \widetilde{p}_{k,q,q^{-1},\gamma}(z), \tag{16}$$

where $\widetilde{p}_{k,q,q^{-1},\gamma}(z)$ is defined by (11).

We also fixed $k\text{-UST}^-(q, q^{-1}, \gamma) = k\text{-UST}(q, q^{-1}, \gamma) \cap \mathcal{T}$, where \mathcal{T} is the subclass of $k\text{-UST}(q, q^{-1}, \gamma)$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, \quad \forall n \geq 2. \tag{17}$$

Our further investigation is organized as follows. In Section 2, we give some supporting results in form of Lemmas, which will help in order to obtain our new results in Section 3. In Section 3, we obtain our main results and also give some of their special cases in the form of Corollaries and Remarks. In Section 4, we conclude our present investigation and also give some future direction to the interested readers toward the prospect that this kind of result will be obtained for other new subclasses of analytic functions.

2. A Set of Lemmas

We need the following lemmas in order to prove our main results.

Lemma 1 (See [45]). Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \prec \mathcal{F}(z) = 1 + \sum_{n=1}^{\infty} d_n z^n$. If $\mathcal{F}(z)$ is convex univalent in E , then

$$|p_n| \leq |d_n|, \quad n \geq 1.$$

Lemma 2. Let $k \in [0, \infty)$ be fixed and

$$\widetilde{p}_{k,q,q^{-1},\gamma}(z) = \frac{2q^{-1}\widetilde{p}_{k,\gamma}(z)}{(q^{-1} + q) + (q^{-1} - q)\widetilde{p}_{k,\gamma}(z)}.$$

Then

$$\widetilde{p}_{k,q,q^{-1},\gamma}(z) = 1 + \frac{2q^{-1}}{q^{-1} + q} Q_1 z + \left\{ \frac{2q^{-1}}{q^{-1} + q} Q_2 - \frac{2(q^{-1} - q)}{q^{-1} + q} Q_1^2 \right\} z^2 + \dots,$$

where Q_1 , and Q_2 is given by (5) and (6).

Proof. From (11), we have

$$\begin{aligned}
 \widetilde{p_{k,q,q^{-1},\gamma}}(z) &= \frac{2q^{-1}\widetilde{p_{k,\gamma}}(z)}{(q^{-1} + q) + (q^{-1} - q)\widetilde{p_{k,\gamma}}(z)} \\
 &= \frac{2q^{-1}}{(q^{-1} + q)} \{\widetilde{p_{k,\gamma}}(z)\} - \frac{2q^{-1}(q^{-1} - q)}{(q^{-1} + q)^2} (\widetilde{p_{k,\gamma}}(z))^2 \\
 &\quad + \frac{2q^{-1}(q^{-1} - q)^2}{(q^{-1} + q)^3} (\widetilde{p_{k,\gamma}}(z))^3 \\
 &\quad - \frac{2q^{-1}(q^{-1} - q)^3}{(q^{-1} + q)^4} (\widetilde{p_{k,\gamma}}(z))^4 + \dots .
 \end{aligned}
 \tag{18}$$

By using (4) in (18), we have

$$\begin{aligned}
 \widetilde{p_{k,q,q^{-1},\gamma}}(z) &= \sum_{n=1}^{\infty} \frac{2q^{-1}(-1)^{n-1}(q^{-1} - q)^{n-1}}{(q^{-1} + q)^n} \\
 &\quad + \sum_{n=1}^{\infty} \frac{2q^{-1}n(-1)^{n-1}(q^{-1} - q)^{n-1}}{(q^{-1} + q)^n} Q_1 z \\
 &\quad + \left\{ \sum_{n=1}^{\infty} \frac{2q^{-1}n(-1)^{n-1}(q^{-1} - q)^{n-1}}{(q^{-1} + q)^n} Q_2 \right. \\
 &\quad \left. - \sum_{n=1}^{\infty} \frac{2q^{-1}(2n - 1)(-1)^{n-1}(q^{-1} - q)^n}{(q^{-1} + q)^{n+1}} Q_1^2 \right\} z^2 + \dots .
 \end{aligned}
 \tag{19}$$

The series $\sum_{n=1}^{\infty} \frac{2q^{-1}(-1)^{n-1}(q^{-1} - q)^{n-1}}{(q^{-1} + q)^n}$, $\sum_{n=1}^{\infty} \frac{2q^{-1}n(-1)^{n-1}(q^{-1} - q)^{n-1}}{(q^{-1} + q)^n}$, and $\sum_{n=1}^{\infty} \frac{2q^{-1}(2n - 1)(-1)^{n-1}(q^{-1} - q)^n}{(q^{-1} + q)^{n+1}}$ are convergent and convergent to 1, $\frac{2q^{-1}}{q^{-1} + q}$, and $\frac{2q^{-1}(q^{-1} - q)}{(q^{-1} + q)}$ respectively.

Therefore, (19) becomes

$$\widetilde{p_{k,q,q^{-1},\gamma}}(z) = 1 + \frac{2q^{-1}}{q^{-1} + q} Q_1 z + \left\{ \frac{2q^{-1}}{q^{-1} + q} Q_2 - \frac{2q^{-1}(q^{-1} - q)}{(q^{-1} + q)} Q_1^2 \right\} z^2 + \dots .
 \tag{20}$$

This complete the proof of Lemma 2. \square

Remark 4. For $q \rightarrow 1-$, Lemma 2 reduces to the lemma introduced by Sim et al. [46].

Lemma 3. Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in k\widetilde{\mathcal{P}}_{q,q^{-1},\gamma}$; then

$$|p_n| \leq \frac{2q^{-1}}{q^{-1} + q} |Q_1|, \quad n \geq 1.$$

Proof. By definition (7), a function $p(z) \in k\widetilde{\mathcal{P}}_{q,q^{-1},\gamma}(z)$ if and only if

$$p(z) \prec \widetilde{p_{k,q,q^{-1},\alpha}}(z),
 \tag{21}$$

where $k \in [0, \infty)$, and $\widetilde{p_{k,q,q^{-1},\alpha}}(z)$ is given by (11).

By using (20) in (21), we have

$$p(z) < 1 + \frac{2q^{-1}}{q^{-1} + q} Q_1 z + \left\{ \frac{2q^{-1}}{q^{-1} + q} Q_2 - \frac{2q^{-1}(q^{-1} - q)}{(q^{-1} + q)} Q_1^2 \right\} z^2 + \dots \tag{22}$$

Now by using Lemma 1 on (22), we have

$$|p_n| \leq \frac{2q^{-1}}{q^{-1} + q} |Q_1|.$$

Hence, the proof of Lemma 3 is complete. \square

Remark 5. For $q \rightarrow 1-$, then Lemma 3 reduces to the lemma introduced by Noor et al. [47].

Lemma 4 ([48]). Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ and $h(z)$ be analytic in E and satisfy $\Re\{h(z)\} > 0$ for z in E . Then,

$$|c_2 - v c_1^2| \leq 2 \max\{1, |2v - 1|\}, \quad \forall v \in \mathbb{C}.$$

3. Main Results

We now state and prove our main results. The first theorem of this section gives necessary conditions for an analytic function f of the form (1) to belong to the newly defined class $k\mathcal{UST}(q, q^{-1}, \gamma)$.

Theorem 1. If an analytic function f is of the form (1), and it satisfies

$$\begin{aligned} & \sum_{n=2}^{\infty} \left\{ 2q^{-1}(k+1) \left| [\widetilde{n}]_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) [\widetilde{n}]_q + 2q^{-1} \right\} \right\} |a_n| \\ & \leq (q + q^{-1}) |\gamma|, \end{aligned} \tag{23}$$

then $f \in k\mathcal{UST}(q, q^{-1}, \gamma)$.

Proof. Suppose that (23) holds; then, it is sufficient to show that

$$\left| \frac{k}{\gamma} \left(\mathcal{J}(q, q^{-1}, f(z)) - 1 \right) \right| - \Re \left\{ \frac{1}{\gamma} \left(\mathcal{J}(q, q^{-1}, f(z)) - 1 \right) \right\} \leq 1.$$

Using (15), we have

$$\begin{aligned}
 & \left| \frac{k}{\gamma} \left(\frac{(q^{-1} + q) \frac{z(\overline{D})_q f(z)}{f(z)}}{(q - q^{-1}) \frac{z(\overline{D})_q f(z)}{f(z)} + 2q^{-1}} - 1 \right) \right| \\
 & - \Re \left\{ \frac{1}{\gamma} \left(\frac{(q^{-1} + q) \frac{z(\overline{D})_q f(z)}{f(z)}}{(q - q^{-1}) \frac{z(\overline{D})_q f(z)}{f(z)} + 2q^{-1}} - 1 \right) \right\} \\
 & \leq \frac{k}{|\gamma|} \left| \frac{(q^{-1} + q) \frac{z(\overline{D})_q f(z)}{f(z)}}{(q - q^{-1}) \frac{z(\overline{D})_q f(z)}{f(z)} + 2q^{-1}} - 1 \right| + \frac{1}{|\gamma|} \left| \frac{(q^{-1} + q) \frac{z(\overline{D})_q f(z)}{f(z)}}{(q - q^{-1}) \frac{z(\overline{D})_q f(z)}{f(z)} + 2q^{-1}} - 1 \right|, \\
 & \leq \frac{(k + 1)}{|\gamma|} \left| \frac{(q^{-1} + q) \frac{z(\overline{D})_q f(z)}{f(z)}}{(q - q^{-1}) \frac{z(\overline{D})_q f(z)}{f(z)} + 2q^{-1}} - 1 \right| \\
 & = \frac{2q^{-1}(k + 1)}{|\gamma|} \left| \frac{\sum_{n=2}^{\infty} (\widetilde{[n]}_q - 1) a_n z^n}{(q + q^{-1}) + \sum_{n=2}^{\infty} \left\{ (q - q^{-1}) \widetilde{[n]}_q + 2q^{-1} \right\} a_n z^n} \right|, \\
 & \leq \frac{2q^{-1}(k + 1)}{|\gamma|} \left\{ \frac{\sum_{n=2}^{\infty} \widetilde{[n]}_q - 1 |a_n|}{(q + q^{-1}) - \sum_{n=2}^{\infty} (q - q^{-1}) \widetilde{[n]}_q + 2q^{-1} |a_n|} \right\}. \tag{24}
 \end{aligned}$$

The expression (41) is bounded above by 1.

$$\frac{2q^{-1}(k + 1)}{|\gamma|} \left\{ \frac{\sum_{n=2}^{\infty} \widetilde{[n]}_q - 1 |a_n|}{(q + q^{-1}) - \sum_{n=2}^{\infty} (q - q^{-1}) \widetilde{[n]}_q + 2q^{-1} |a_n|} \right\} < 1.$$

After some simple calculations, we have

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \left\{ 2q^{-1}(k + 1) \left| \widetilde{[n]}_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) \widetilde{[n]}_q \right| + 2q^{-1} \right\} \right\} |a_n| \\
 & \leq (q + q^{-1}) |\gamma|.
 \end{aligned}$$

Hence, we complete the proof of Theorem 1. \square

When $q \rightarrow 1-$ and $\gamma = 1 - \alpha$ with $0 \leq \alpha < 1$, we have the following known result (see [7]).

Corollary 1. An analytic function f of the form (1) belongs to the class $k\text{-UST}(\alpha)$ if it satisfies the condition

$$\sum_{n=2}^{\infty} \{n(k + 1) - (k + \alpha)\} |a_n| \leq 1 - \alpha,$$

where $0 \leq \alpha < 1$ and $k \geq 0$.

Inequality (23) gives the following corollary:

Corollary 2. Let $0 \leq k < \infty$, $q \in (0, 1)$ and $\gamma \in \mathbb{C} \setminus \{0\}$. If the inequality

$$|a_n| \leq \frac{(q + q^{-1}) |\gamma|}{\left\{ 2q^{-1}(k + 1) \left| \widetilde{[n]}_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) \widetilde{[n]}_q \right| + 2q^{-1} \right\} \right\}}, \quad n \geq 2,$$

holds for $f(z) = z + a_n z^n$, then $k\text{-UST}(q, q^{-1}, \gamma)$. In particular,

$$f(z) = z + \frac{(q + q^{-1})|\gamma|}{\left\{2q^{-1}(k + 1)\left[\widetilde{[2]}_q - 1\right] + |\gamma|\left\{\left|(q - q^{-1})\left[\widetilde{[2]}_q\right] + 2q^{-1}\right\}\right\}} z^2 \in k\text{-UST}(q, q^{-1}, \gamma)$$

and

$$|a_2| = \frac{(q + q^{-1})|\gamma|}{\left\{2q^{-1}(k + 1)\left[\widetilde{[2]}_q - 1\right] + |\gamma|\left\{\left|(q - q^{-1})\left[\widetilde{[2]}_q\right] + 2q^{-1}\right\}\right\}}.$$

Coefficient estimates for the class $k\text{-UST}(q, q^{-1}, \gamma)$ are given in the next Theorem.

Theorem 2. If $f \in k\text{-UST}(q, q^{-1}, \gamma)$ and is of the form (1), then

$$|a_2| \leq \frac{2q^{-1}|Q_1|}{(q^{-1} + q)} \tag{25}$$

and

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{|Q_1 - \widetilde{[j]}_q|}{(q^{-1} + q)\widetilde{[j + 1]}_q} \right) \varphi_j \text{ for } n \geq 3, \tag{26}$$

where Q_1 and φ_j are defined by (5) and (29).

Proof. Let

$$\frac{(q^{-1} + q) \frac{z(\widetilde{D})_q f(z)}{f(z)}}{(q - q^{-1}) \frac{z(\widetilde{D})_q f(z)}{f(z)} + 2q^{-1}} = p(z). \tag{27}$$

After some simple simplification of Equation (27), we have

$$\begin{aligned} & (q^{-1} + q)z + \sum_{n=2}^{\infty} (q^{-1} + q)\widetilde{[n]}_q a_n z^n \\ & = \left\{ (q^{-1} + q)z + \sum_{n=2}^{\infty} \left((q - q^{-1})\widetilde{[n]}_q + 2q^{-1} \right) a_n z^n \right\} \left(1 + \sum_{n=1}^{\infty} c_n z^n \right). \end{aligned}$$

Equating coefficients of z^n on both sides, we have

$$2q^{-1}(\widetilde{[n]}_q - 1)a_n = \sum_{j=1}^{n-1} (\widetilde{[j - 1]}_q (q - q^{-1}) + 2q^{-1}) a_{n-j} c_j, \quad a_1 = 1.$$

This implies that

$$|a_n| \leq \frac{1}{2q^{-1}(\widetilde{[n]}_q - 1)} \sum_{j=1}^{n-1} \left\{ \widetilde{[j - 1]}_q (q - q^{-1}) + 2q^{-1} \right\} |a_{n-j}| |c_j|.$$

By using Lemma 3, we have

$$\begin{aligned} |a_n| & \leq \frac{|Q_1|}{(q^{-1} + q)(\widetilde{[n]}_q - 1)} \sum_{j=1}^{n-1} \left\{ \widetilde{[j - 1]}_q (q - q^{-1}) + 2q^{-1} \right\} |a_j|, \\ |a_n| & \leq \frac{|Q_1|}{(q^{-1} + q)(\widetilde{[n]}_q - 1)} \sum_{j=1}^{n-1} \varphi_{j-1} |a_j|, \end{aligned} \tag{28}$$

where

$$\varphi_{j-1} = \widetilde{[j - 1]}_q (q - q^{-1}) + 2q^{-1}. \tag{29}$$

Now we prove that

$$\frac{|Q_1|}{(q^{-1} + q)([n]_q - 1)} \sum_{j=1}^{n-1} \varphi_{j-1} |a_j| \leq \prod_{j=0}^{n-2} \left(\frac{|Q_1 - [j]_q|}{(q^{-1} + q)([j+1]_q)} \right) \varphi_j. \tag{30}$$

For this, we use the induction method. For $n = 2$ from (28), we have

$$|a_2| \leq \frac{2q^{-1}|Q_1|}{(q^{-1} + q)}, \quad \varphi_0 = 2q^{-1}$$

From (26), we have

$$|a_2| \leq \frac{2q^{-1}|Q_1|}{(q^{-1} + q)}, \quad \varphi_0 = 2q^{-1}.$$

For $n = 3$, from (28), we have

$$\begin{aligned} |a_3| &\leq \frac{|Q_1|}{(q^{-1} + q)([3]_q - 1)} (\varphi_0 + \varphi_1 |a_2|), \\ &\leq \frac{2q^{-1}|Q_1|}{(q^{-1} + q)([3]_q - 1)} (1 + |Q_1|), \quad \varphi_1 = (q^{-1} + q). \end{aligned}$$

From (26), we have

$$\begin{aligned} |a_3| &\leq \frac{|Q_1|\varphi_0}{(q^{-1} + q)} \left\{ \left(\frac{|Q_1 - [1]_q|}{(q^{-1} + q)[2]_q} \right) \varphi_1 \right\}, \\ &\leq \frac{|Q_1|\varphi_0}{(q^{-1} + q)} \left\{ \left(\frac{|Q_1| + [1]_q}{(q^{-1} + q)[2]_q} \right) \varphi_1 \right\}, \\ &= \frac{|Q_1|\varphi_1}{(q^{-1} + q)[2]_q} \left(\frac{|Q_1|\varphi_0}{(q^{-1} + q)} + \frac{\varphi_0}{(q^{-1} + q)} \right), \\ &= \frac{|Q_1|\varphi_1}{(q^{-1} + q)[2]_q} \left(\frac{|Q_1|\varphi_0}{(q^{-1} + q)} + \frac{\varphi_0}{(q^{-1} + q)} \right), \\ &= \frac{2q^{-1}|Q_1|}{(q^{-1} + q)[2]_q} (|Q_1| + 1). \end{aligned}$$

Let the hypothesis be true for $n = m$. From (28), we have

$$|a_m| \leq \frac{|Q_1|}{(q^{-1} + q)([m]_q - 1)} \sum_{j=1}^{m-1} \varphi_{j-1} |a_j|.$$

From (26), we have

$$\begin{aligned} |a_m| &\leq \prod_{j=0}^{m-2} \left(\frac{|Q_1 - [j]_q|}{(q^{-1} + q)[j+1]_q} \right) \varphi_j, \quad n \geq 2, \\ &\leq \prod_{j=0}^{m-2} \left(\frac{|Q_1| + [j]_q}{(q^{-1} + q)[j+1]_q} \right) \varphi_j, \quad n \geq 2. \end{aligned}$$

By the induction hypothesis, we have

$$\frac{|Q_1|}{(q^{-1} + q)(\widetilde{[m]}_q - 1)} \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \leq \prod_{j=0}^{m-2} \left(\frac{|Q_1| + \widetilde{[j]}_q}{(q^{-1} + q)\widetilde{[j+1]}_q} \right) \varphi_j. \tag{31}$$

Multiplying $\frac{|Q_1| + (\widetilde{[m]}_q - 1)}{(q^{-1} + q)(\widetilde{[m]}_q - 1)}$ on both sides of (31), we have

$$\begin{aligned} & \prod_{j=0}^{m-2} \left(\frac{|Q_1| + \widetilde{[j]}_q}{(q^{-1} + q)\widetilde{[j+1]}_q} \right) \varphi_j, \\ & \geq \frac{|Q_1| + (\widetilde{[m]}_q - 1)}{(q^{-1} + q)(\widetilde{[m]}_q - 1)} \left\{ \frac{|Q_1|}{(q^{-1} + q)(\widetilde{[m]}_q - 1)} \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \right\}, \\ & = \frac{|Q_1|}{(q^{-1} + q)(\widetilde{[m]}_q - 1)} \left\{ \frac{|Q_1|}{(q^{-1} + q)(\widetilde{[m]}_q - 1)} + 1 \right\} \sum_{j=1}^{m-1} \varphi_{j-1} |a_j|, \\ & \geq \frac{|Q_1|}{(q^{-1} + q)(\widetilde{[m]}_q - 1)_q} \left\{ |a_m| + \sum_{j=1}^{m-1} \varphi_{j-1} |a_j| \right\}, \\ & = \frac{|Q_1|}{(q^{-1} + q)(\widetilde{[m]}_q - 1)} \sum_{j=1}^m \varphi_{j-1} |a_j|. \end{aligned}$$

That is

$$\frac{|Q_1|}{(q^{-1} + q)(\widetilde{[m]}_q - 1)} \sum_{j=1}^m \varphi_{j-1} |a_j| \leq \prod_{j=0}^{m-2} \left(\frac{|Q_1| + \widetilde{[j]}_q}{(q^{-1} + q)\widetilde{[j+1]}_q} \right) \varphi_j,$$

which shows that inequality (31) is true for $n = m + 1$. Hence, the proof of Theorem 2 is now completed. \square

Corollary 3. If $f(z) \in k\text{-UST}(\gamma)$ and is of the form (1), then

$$|a_n| \leq \prod_{j=0}^{n-2} \left(\frac{|Q_1 - j|}{(j + 1)} \right) \text{ for } n \geq 3.$$

For $q \rightarrow 1-$, Kanas and Wisniowska proved Corollary (3) in [4].

In the next Theorem, we state and prove the Fekete–Szegő-type result for our defined function class $k\text{-UST}(q, q^{-1}, \gamma)$.

Theorem 3. Let $0 \leq k < \infty, q \in (0, 1)$ be fixed, and let $f(z) \in k\text{-UST}(q, q^{-1}, \gamma)$ and be of the form (1). Then for a complex number μ ,

$$|a_3 - \mu a_2^2| \leq \frac{|Q_1|}{2(q^{-1} + q)(\widetilde{[3]}_q - 1)} \max\{1, |2v - 1|\}, \tag{32}$$

where v is given by (37).

Proof. If $f(z) \in k\text{-UST}(q, q^{-1}, \gamma)$, then we have

$$\mathcal{J}(q, q^{-1}, f(z)) \prec \widetilde{p_{k,q,q^{-1},\gamma}}(z)$$

and if there exists a Schwarz function $w(z)$, we have

$$\frac{(q^{-1} + q) \frac{zD_q f(z)}{f(z)}}{(q - q^{-1}) \frac{zD_q f(z)}{f(z)} + 2q^{-1}} = p_{k,q,q^{-1},\gamma}(w(z)). \tag{33}$$

Let the function $h(z) \in \mathcal{P}$, defined by:

$$h(z) = \frac{1 + w(z)}{1 - w(z)},$$

this gives

$$w(z) = \frac{c_1}{2}z + \frac{1}{2} \left(c_2 - \frac{c_1^2}{2} \right) z^2 + \dots$$

and

$$p_{k,q,q^{-1},\gamma}(w(z)) = 1 + \frac{q^{-1}Q_1c_1}{(q^{-1} + q)}z + \frac{q^{-1}}{(q^{-1} + q)}F_1(q, q^{-1})z^2 + \dots, \tag{34}$$

where

$$F_1(q, q^{-1}) = \left\{ \frac{Q_2c_1^2}{2} + (c_2 - \frac{c_1^2}{2})Q_1 - \frac{(q^{-1} - q)Q_1^2c_1^2}{2} \right\}.$$

Now, from L.H.S of (33), we have

$$\frac{(q^{-1} + q) \frac{zD_q f(z)}{f(z)}}{(q - q^{-1}) \frac{zD_q f(z)}{f(z)} + 2q^{-1}} = 1 + \frac{2q^{-1}([\widetilde{2}]_q - 1)}{(q^{-1} + q)}a_2z + F_2(q, q^{-1})z^2 + \dots, \tag{35}$$

where

$$F_2(q, q^{-1}) = \left\{ 2q^{-1}([\widetilde{3}]_q - 1)a_3 - \frac{2q^{-1}}{(q^{-1} + q)}([\widetilde{2}]_q - 1) \left\{ (q - q^{-1})[\widetilde{2}]_q + 2q^{-1} \right\} a_2^2 \right\}.$$

By using (34) and (35) in (33), we obtain

$$a_2 = \frac{Q_1c_1}{2([\widetilde{2}]_q - 1)}$$

and

$$a_3 = \frac{1}{2(q^{-1} + q)([\widetilde{3}]_q - 1)} \left\{ F_1(q, q^{-1}) + \frac{\left\{ (q - q^{-1})[\widetilde{2}]_q + 2q^{-1} \right\} Q_1^2c_1^2}{2([\widetilde{2}]_q - 1)} \right\}.$$

For any complex number μ we have

$$|a_3 - \mu a_2^2| = \frac{|Q_1|}{2(q^{-1} + q)([\widetilde{3}]_q - 1)} \left\{ c_2 - v c_1^2 \right\}, \tag{36}$$

where

$$v = \frac{1}{2} \left\{ 1 - \frac{Q_2}{Q_1} + (q^{-1} - q)Q_1 - F_3(q, q^{-1}) \right\} \tag{37}$$

and

$$F_3(q, q^{-1}) = \frac{\left\{ (q - q^{-1})[\widetilde{2}]_q + 2q^{-1} \right\} Q_1}{([\widetilde{2}]_q - 1)} + \mu \frac{(q^{-1} + q)([\widetilde{3}]_q - 1)Q_1}{([\widetilde{2}]_q - 1)^2}.$$

Now by using Lemma 4 on (36), we have

$$|a_3 - \mu a_2^2| \leq \frac{|Q_1|}{2(q^{-1} + q)([\widetilde{3}]_q - 1)} \max\{1, |2v - 1|\}.$$

Hence, we complete the proof of Theorem 3. \square

Next we investigate the necessary and sufficient conditions for $f(z)$ of the form (17) to be in the class $k\text{-UST}^-(q, q^{-1}, \gamma)$.

Theorem 4. Let $k \in [0, \infty)$, $q \in (0, 1)$ and $\gamma \in \mathbb{C} \setminus \{0\}$. A function $f(z)$ of the form (17) will belong to the class $k\text{-UST}^-(q, q^{-1}, \gamma)$ which can be expressed as:

$$\begin{aligned} & \sum_{n=2}^{\infty} \left\{ 2q^{-1}(k+1) \left| [\widetilde{n}]_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) [\widetilde{n}]_q \right| + 2q^{-1} \right\} \right\} |a_n| \\ & \leq (q + q^{-1}) |\gamma|. \end{aligned} \tag{38}$$

The result is sharp for the function

$$f(z) = z - \frac{(q + q^{-1}) |\gamma|}{2q^{-1}(k+1) \left| [\widetilde{n}]_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) [\widetilde{n}]_q \right| + 2q^{-1} \right\}} z^n.$$

Proof. In view of Theorem 1, it remains to prove the necessity.

If $f(z) \in k\text{-UST}^-(q, q^{-1}, \gamma)$, then infer that $|\Re(z)| \leq |z|$; for any z , we have

$$\begin{aligned} & \left| 1 + \frac{1}{\gamma(q + q^{-1})} \left(\frac{\sum_{n=2}^{\infty} 2q^{-1}([\widetilde{n}]_q - 1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{1}{q+q^{-1}} \left\{ (q - q^{-1}) [\widetilde{n}]_q + 2q^{-1} \right\} a_n z^{n-1}} \right) \right| \\ & \geq \left| \frac{k}{\gamma(q + q^{-1})} \left\{ \frac{\sum_{n=2}^{\infty} 2q^{-1}([\widetilde{n}]_q - 1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{1}{(q+q^{-1})} \left\{ (q - q^{-1}) [\widetilde{n}]_q + 2q^{-1} \right\} a_n z^{n-1}} \right\} \right|. \end{aligned} \tag{39}$$

Letting $z \rightarrow 1^-$ along the real axis, we obtain the desired inequality (38). Hence we complete the proof of Theorem 4. \square

Corollary 4. Let the function $f(z)$ of the form (17) be in the class $k\text{-UST}^-(q, q^{-1}, \gamma)$. Then

$$a_n \leq \frac{(q + q^{-1}) |\gamma|}{2q^{-1}(k+1) \left| [\widetilde{n}]_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) [\widetilde{n}]_q \right| + 2q^{-1} \right\}}, \quad n \geq 2. \tag{40}$$

Corollary 5. Let the function $f(z)$ of the form (17) be in the class $k\text{-UST}^-(q, q^{-1}, \gamma)$. Then

$$a_2 = \frac{(q + q^{-1}) |\gamma|}{2q^{-1}(k+1) \left| [\widetilde{2}]_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) [\widetilde{2}]_q \right| + 2q^{-1} \right\}}. \tag{41}$$

Theorem 5. Let $k \in [0, \infty)$, $q \in (0, 1)$ and $\gamma \in \mathbb{C} \setminus \{0\}$ and let

$$f_1(z) = z,$$

and

$$f_n(z) = z - \mathcal{R}(n, q) z^n, \quad n \geq 3. \tag{42}$$

Then $f \in k\mathcal{UST}^-(q, q^{-1}, \gamma)$, if and only if f can be expressed in the form of

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z), \quad \lambda_n > 0, \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = 1, \tag{43}$$

where $\mathcal{R}(n, q)$ is given by (44).

Proof. Suppose that

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \lambda_n f_n(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z), \\ &= \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n \{z - \mathcal{R}(n, q)z^n\}, \\ &= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n z - \sum_{n=2}^{\infty} \lambda_n \mathcal{R}(n, q)z^n, \\ &= \left(\sum_{n=1}^{\infty} \lambda_n\right)z - \sum_{n=2}^{\infty} \lambda_n \mathcal{R}(n, q)z^n, \\ &= z - \sum_{n=2}^{\infty} \lambda_n \mathcal{R}(n, q)z^n, \end{aligned}$$

where

$$\mathcal{R}(n, q) = \frac{(q + q^{-1})|\gamma|}{2q^{-1}(k + 1) \left| \widetilde{[n]}_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1})\widetilde{[n]}_q \right| + 2q^{-1} \right\}}. \tag{44}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty} \lambda_n \mathcal{R}(n, q) \times \frac{1}{\mathcal{R}(n, q)} \\ &= \sum_{n=2}^{\infty} \lambda_n = \sum_{n=1}^{\infty} \lambda_n - \lambda_1 = 1 - \lambda_1 \leq 1, \end{aligned}$$

and we find $k\mathcal{UST}^-(q, q^{-1}, \gamma)$.

Conversely, suppose that $k\mathcal{UST}^-(q, q^{-1}, \gamma)$. Since

$$|a_n| \leq \mathcal{R}(n, q),$$

we can set

$$\lambda_n = \frac{1}{\mathcal{R}(n, q)} |a_n|,$$

and

$$\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

Then

$$\begin{aligned}
 f(z) &= z + \sum_{n=2}^{\infty} a_n z^n \\
 &= z + \sum_{n=2}^{\infty} \lambda_n \mathcal{R}(n, q) z^n, \\
 &= z + \sum_{n=2}^{\infty} \lambda_n (z + f_n(z)) = z + \sum_{n=2}^{\infty} \lambda_n z + \sum_{n=2}^{\infty} \lambda_n f_n(z), \\
 &= \left(1 - \sum_{n=2}^{\infty} \lambda_n\right) z + \sum_{n=2}^{\infty} \lambda_n f_n(z) = \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n f_n(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z).
 \end{aligned}$$

The proof of Theorem 5 is complete. \square

Theorem 6. Let $k \in [0, \infty)$, $q \in (0, 1)$, $|z| = r < 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$. Let f of the form (17) be in the class $k - \mathcal{UST}^-(q, q^{-1}, \gamma)$. Then

$$r - \mathcal{R}(2, q)r^2 \leq |f(z)| \leq r + \mathcal{R}(2, q)r^2. \tag{45}$$

Equality in (45) is attained for the function f given by the formula

$$f(z) = z + \mathcal{R}(2, q)z^2, \tag{46}$$

where

$$\mathcal{R}(2, q) = \frac{(q + q^{-1})|\gamma|}{\left\{2q^{-1}(k + 1)\left|\widetilde{[2]}_q - 1\right| + |\gamma|\left\{\left|(q - q^{-1})\widetilde{[2]}_q\right| + 2q^{-1}\right\}\right\}}.$$

Proof. Since $f \in k - \mathcal{UST}^-(q, q^{-1}, \gamma)$, in view of Theorem 4, we find

$$\begin{aligned}
 &\left\{2q^{-1}(k + 1)\left|\widetilde{[2]}_q - 1\right| + |\gamma|\left\{\left|(q - q^{-1})\widetilde{[2]}_q\right| + 2q^{-1}\right\}\right\} \sum_{n=2}^{\infty} a_n \\
 &\leq \sum_{n=2}^{\infty} \left\{2q^{-1}(k + 1)\left|\widetilde{[n]}_q - 1\right| + |\gamma|\left\{\left|(q - q^{-1})\widetilde{[n]}_q\right| + 2q^{-1}\right\}\right\} |a_n| \\
 &\leq (q + q^{-1})|\gamma|.
 \end{aligned}$$

This gives

$$\sum_{n=2}^{\infty} a_n \leq \mathcal{R}(2, q).$$

Therefore

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq r + \mathcal{R}(2, q)r^2,$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n \geq r - \mathcal{R}(2, q)r^2.$$

By letting $r \rightarrow 1^-$, we get the required result. Hence, the proof of Theorem 6 is complete. \square

Theorem 7. Let $k \in [0, \infty)$, $q \in (0, 1)$, $|z| = r < 1$ and $\gamma \in \mathbb{C} \setminus \{0\}$. Let f be of the form (17) in the class $k - \mathcal{UST}^-(q, q^{-1}, \gamma)$. Then

$$1 - 2\mathcal{R}(2, q)r \leq |f'(z)| \leq 1 + 2\mathcal{R}(2, q)r. \tag{47}$$

Proof. Differentiating f and using triangle inequality for the modulus, we obtain

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} na_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} na_n, \tag{48}$$

and

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} na_n |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} na_n. \tag{49}$$

Assertion (47) follows from (48) and (49); in view of the rather simple consequence, we get the following inequality

$$\sum_{n=2}^{\infty} na_n \leq 2\mathcal{R}(2, q).$$

Hence, proof of Theorem 7 is complete. \square

Theorem 8. Under convex linear combination, the class $k\text{-}\mathcal{UST}^-(q, q^{-1}, \gamma)$ is closed.

Proof. Let $f(z)$ and $g(z)$ be in class $k\text{-}\mathcal{UST}^-(q, q^{-1}, \gamma)$. Suppose $f(z)$ is given by (17) and

$$g(z) = z - \sum_{n=2}^{\infty} d_n z^n, \tag{50}$$

where $a_n, d_n \geq 0$.

It is sufficient to show that for $0 \leq \lambda \leq 1$, the function

$$\mathcal{H}(z) = \lambda f(z) + (1 - \lambda)g(z) \in k\text{-}\mathcal{UST}^-(q, q^{-1}, \gamma). \tag{51}$$

From (17), (50) and (51), we have

$$\mathcal{H}(z) = z - \sum_{n=2}^{\infty} \{\lambda a_n + (1 - \lambda)d_n\}z^n. \tag{52}$$

As we know that f and g are in class $k\text{-}\mathcal{UST}^-(q, q^{-1}, \gamma)$ ($0 \leq \lambda \leq 1$), by using Theorem 4, we obtain

$$\begin{aligned} & \sum_{n=2}^{\infty} \left\{ 2q^{-1}(k+1) \left| [\widetilde{n}]_q - 1 \right| + |\gamma| \left\{ \left| (q - q^{-1}) [\widetilde{n}]_q \right| + 2q^{-1} \right\} \right\} \{\lambda a_n + (1 - \lambda)d_n\} \\ & \leq (1 + q)|\gamma|. \end{aligned} \tag{53}$$

Again, by Theorem 4 and inequality (53), we have $\mathcal{H}(z) \in k\text{-}\mathcal{UST}^-(q, q^{-1}, \gamma)$. Hence, the proof of Theorem 8 is complete. \square

4. Conclusions

In this paper, motivated significantly by a number of recent works, we used the concept of symmetric quantum calculus and conic regions to define a new domain $\widetilde{\Omega}_{k,q,q^{-1},\gamma}$, which generalizes the symmetric conic domains. By using a certain generalized symmetric conic domain $\widetilde{\Omega}_{k,q,q^{-1},\gamma}$, we defined and investigated a new subclass of normalized analytic q -starlike functions in the open unit disk E , and we have successfully derived several properties and characteristics of a newly defined subclass of analytic functions. For the verification and validity of our main results, we have also pointed out relevant connections of our main results with those in several earlier related works on this subject.

To conclude our present investigation, we would like to remark that one may attempt the results presented in this paper for different subclasses of analytic functions in different domains. In particular, one may define a new subclass of symmetric q -starlike functions associated with this newly defined domain and can obtain the same results.

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References

1. Goodman, A.W. *Univalent Functions*; Polygonal Publishing House: Washington, NJ, USA, 1983; Volume I.
2. Goodman, A.W. *Univalent Functions*; Polygonal Publishing House: Washington, NJ, USA, 1983; Volume II.
3. Goodman, A.W. On uniformly convex functions. *Ann. Polon. Math.* **1991**, *56*, 87–92. [[CrossRef](#)]
4. Kanas, S.; Wisniowska, A. Conic domains and k -starlike functions. *Rev. Roum. Math. Pure Appl.* **2000**, *45*, 647–657.
5. Rønning, F. Uniformly convex functions and a corresponding class of starlike functions. *Proc. Am. Math. Soc.* **1993**, *118*, 189–196. [[CrossRef](#)]
6. Ma, W.; Minda, D. Uniformly convex functions. *Ann. Polon. Math.* **1992**, *57*, 165–175. [[CrossRef](#)]
7. Shams, S.; Kulkarni, S.R.; Jahangiri, J.M. Classes of uniformly starlike and convex functions. *Int. J. Math. Math. Sci.* **2004**, *55*, 2959–2961. [[CrossRef](#)]
8. Jackson, F.H. On q -functions and a certain difference operator. *Earth Environ. Sci. Trans. R. Edinb.* **1909**, *46*, 253–281. [[CrossRef](#)]
9. Islam, S.; Khan, M.G.; Ahmad, B.; Arif, M.; Chinram, R. q -Extension of Starlike Functions Subordinated with a Trigonometric Sine Function. *Mathematics* **2020**, *8*, 1676. [[CrossRef](#)]
10. Zainab, S.; Raza, M.; Xin, Q.; Jabeen, M.; Malik, S.N.; Riaz, S. On q -Starlike Functions Defined by q -Ruscheweyh Differential Operator in Symmetric Conic Domain. *Symmetry* **2021**, *13*, 1947. [[CrossRef](#)]
11. Jackson, F.H. On q -definite integrals. *Q. J. Pure Appl. Math.* **1910**, *41*, 193–203.
12. Khan, S.; Hussain, S.; Zaighum, M.A.; Darus, M. A subclass of uniformly convex functions and a corresponding subclass of starlike function with fixed coefficient associated with q -analogue of Ruscheweyh operator. *Math. Slovaca* **2019**, *69*, 825–832. [[CrossRef](#)]
13. Mahmood, S.; Khan, I.; Srivastava, H.M.; Malik, S.N. Inclusion relations for certain families of integral operators associated with conic regions. *J. Inequalities Appl.* **2019**, *2019*, 59. [[CrossRef](#)]
14. Mahmood, S.; Jabeen, M.; Malik, S.N.; Srivastava, H.M.; Manzoor, R.; Riaz, S.M. Some coefficient inequalities of q -starlike functions associated with conic domain defined by q -derivative. *J. Funct. Spaces* **2018**, *2018*, 8492072. [[CrossRef](#)]
15. Ahmad, Q.Z.; Khan, N.; Raza, M.; Tahir, M.; Khan, B. Certain q -difference operators and their applications to the subclass of meromorphic q -starlike functions. *Filomat* **2019**, *33*, 3385–3397. [[CrossRef](#)]
16. Rehman, M.S.; Ahmad, Q.Z.; Khan, B.; Tahir, M.; Khan, N. Generalisation of certain subclasses of analytic and univalent functions, Maejo Internat. *J. Sci. Technol.* **2019**, *13*, 1–9.
17. Shi, L.; Ahmad, B.; Khan, N.; Khan, M.G.; Araci, S.; Mashwani, W.K.; Khan, B. Coefficient Estimates for a Subclass of Meromorphic Multivalent q -Close-to-Convex Functions. *Symmetry* **2021**, *13*, 1840. [[CrossRef](#)]
18. Uçar, H.E.O. Coefficient inequality for q -starlike Functions. *Appl. Math. Comput.* **2016**, *76*, 122–126.
19. Srivastava, H.M. Operators of basic (or q -) calculus and fractional q -calculus and their applications in geometric function theory of complex analysis. *Iran. J. Sci. Technol. Trans. Sci.* **2020**, *44*, 327–344. [[CrossRef](#)]
20. Srivastava, H.M. Univalent functions, fractional calculus, and associated generalized hypergeometric functions. In *Univalent functions, fractional Calculus, and Their Applications*; Srivastava, H.M., Owa, S., Eds.; John Wiley & Sons: New York, NY, USA, 1989.
21. Khan, B.; Liu, Z.-G.; Srivastava, H.M.; Khan, N.; Tahir, M. Applications of higher-order derivatives to subclasses of multivalent q -starlike functions. *Maejo Int. J. Sci. Technol.* **2021**, *15*, 61–72.
22. Srivastava, H.M.; Ahmad, Q.Z.; Khan, N.; Khan, N.; Khan, B. Hankel and Toeplitz determinants for a subclass of q -starlike functions associated with a general conic domain. *Mathematics* **2019**, *7*, 181. [[CrossRef](#)]
23. Ahmad, B.; Khan, M.G.; Frasin, B.A.; Aouf, M.K.; Abdeljawad, T.; Mashwani, W.K.; Arif, M. On q -analogue of meromorphic multivalent functions in lemniscate of Bernoulli domain. *AIMS Math.* **2021**, *6*, 3037–3052. [[CrossRef](#)]
24. Ahmad, B.; Khan, M.G.; Darus, M.; Khan Mashwani, W.; Arif, M. Applications of Some Generalized Janowski Meromorphic Multivalent Functions. *J. Math.* **2021**, *2021*, 6622748. [[CrossRef](#)]
25. Khan, N.; Srivastava, H.M.; Rafiq, A.; Arif, M.; Arjika, S. Some applications of q -difference operator involving a family of meromorphic harmonic functions. *Adv. Differ. Equ.* **2021**, *2021*, 471. [[CrossRef](#)]
26. Zhan, C.; Khan, S.; Hussain, A.; Khan, N.; Hussain, S.; Khan, N. Applications of q -difference symmetric operator in harmonic univalent functions. *AIMS Math.* **2021**, *7*, 667–680. [[CrossRef](#)]

27. Khan, S.; Hussain, S.; Naeem, M.; Darus, M.; Rasheed, A. A Subclass of q -starlike functions defined by using a symmetric q -derivative operator and related with generalized symmetric conic domains. *Mathematics* **2021**, *9*, 917. [[CrossRef](#)]
28. Khan, B.; Liu, Z.-G.; Shaba, T.G.; Araci, S.; Khan, N.; Khan, M.G. Applications of q -Derivative Operator to the Subclass of Bi-Univalent Functions Involving q -Chebyshev Polynomials. *J. Math.* **2022**, *2022*, 8162182. [[CrossRef](#)]
29. Cruz, A.M.D.; Martins, N. The q -symmetric variational calculus. *Comput. Math. Appl.* **2012**, *64*, 2241–2250. [[CrossRef](#)]
30. Lavagno, A. Basic-deformed quantum mechanics. *Rep. Math. Phys* **2009**, *64*, 79–88. [[CrossRef](#)]
31. Gasper, G.; Rahman, M. Volume 35 of Encyclopedia of Mathematics and its applications. In *Basic Hypergeometric Series*; Ellis Horwood: Chichester, UK, 1990.
32. Kamel, B.; Yosr, S. On some symmetric q -special functions. *Le Matematiche* **2013**, *68*, 107–122.
33. Kanas, S.; Raducanu, D. Some class of analytic functions related to conic domains. *Math. Slovaca* **2014**, *64*, 1183–1196. [[CrossRef](#)]
34. Kanas, S.; Wisniowska, A. Conic regions and k -uniform convexity. *J. Comput. Appl. Math.* **1999**, *105*, 327–336. [[CrossRef](#)]
35. Noor, K.I. Applications of certain operators to the classes related with generalized Janowski functions. *Integral Transform. Spec. Funct.* **2010**, *21*, 557–567. [[CrossRef](#)]
36. Noor, K.I.; Malik, S.N. On coefficient inequalities of functions associated with conic domains. *Comput. Math. Appl.* **2011**, *62*, 2209–2217. [[CrossRef](#)]
37. Noor, K.I.; Malik, S.N.; Arif, M.; Raza, M. On bounded boundary and bounded radius rotation related with Janowski function. *World Appl. Sci. J.* **2011**, *12*, 895–902.
38. Sokół, J. Classes of multivalent functions associated with a convolution operator. *Comput. Math. Appl.* **2010**, *60*, 1343–1350. [[CrossRef](#)]
39. Zhang, X.; Khan, S.; Hussain, S.; Tang, H.; Shareef, Z. New subclass of q -starlike functions associated with generalized conic domain. *AIMS Math.* **2020**, *5*, 4830–4848. [[CrossRef](#)]
40. Andrei, L.; Caus, V.-A. A Generalized Class of Functions Defined by the q -Difference Operator. *Symmetry* **2021**, *13*, 2361. [[CrossRef](#)]
41. Cătaș, A.A. On the Fekete–Szegő Problem for Meromorphic Functions Associated with p, q -Wright Type Hypergeometric Function. *Symmetry* **2021**, *13*, 2143. [[CrossRef](#)]
42. Swamy, S.R.; Alb Lupaș, A. Bi-univalent Function Subfamilies Defined by q -Analogue of Bessel Functions Subordinate to (p, q) -Lucas Polynomials. *WSEAS Trans. Math.* **2022**, *21*, 98–106. [[CrossRef](#)]
43. Akgül, A.; Cofîrlă, L.-I. Coefficient Estimates for a Family of Starlike Functions Endowed with Quasi Subordination on Conic Domain. *Symmetry* **2022**, *14*, 582. [[CrossRef](#)]
44. Janowski, W. Some extremal problems for certain families of analytic functions. *Ann. Polon. Math.* **1973**, *28*, 297–326. [[CrossRef](#)]
45. Rogosinski, W. On the coefficients of subordinate functions. *Proc. Lond. Math. Soc* **1943**, *48*, 48–82. [[CrossRef](#)]
46. Sim, Y.J.; Kwon, O.S.; Cho, N.E.; Srivastava, H.M. Some classes of analytic functions associated with conic regions. *Taiwan J. Math.* **2012**, *16*, 387–408. [[CrossRef](#)]
47. Noor, K.I.; Arif, M.; Ul-Haq, W. On k -uniformly close-to-convex functions of complex order. *Appl. Math. Comput.* **2009**, *215*, 629–635. [[CrossRef](#)]
48. Ma, W.; Minda, D. A unified treatment of some special classes of univalent functions. In *Proceedings of the Conferene on Complex Analysis*; Li, Z., Ren, F., Yang, L., Zhang, S., Eds.; International Press Inc., New York, 1992; pp. 157–169.