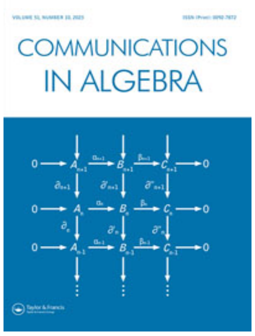

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Quasi-*S*-primary ideals of commutative rings

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ABSTRACT

Let R be a commutative ring with $1 \neq 0$ and S be a multiplicatively closed subset of R. We call an ideal I of R disjoint with S quasi-S-primary if there exists an $s \in S$ such that whenever $a, b \in R$ and $ab \in I$, then $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. We investigate many properties and characterizations of quasi-S-primary ideals. We discuss the form of quasi-S-primary ideals in polynomial, power series, the Serre's conjecture and the Nagata rings. Futhermore, we study quasi-S-primary ideals in amalgamated algebras. Our results allow us to construct original examples of quasi-S-primary ideals.

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1. Introduction

Throughout this article all rings are commutative with non-zero identity. Unless otherwise stated, R denotes a ring and S is a multiplicatively closed subset of R. This article is motivated by the interesting results proved on S-primary ideals of a commutative ring in [20]. According to [20] an ideal I of R with $I \cap S = \emptyset$ is said to be an S-*primary* ideal of R if there exists $s \in S$ such that for all $a, b \in R$ with $ab \in I$, we have either $sa \in I$ or $sb \in \sqrt{I}$. In [20], S. Visweswaran proved the S-version of several classical results on primary ideals. Let R be a commutative ring, S a multiplicatively closed subset of R and I an ideal of R disjoint from S. It was shown in [20, Theorem 2.7] that I is an S-primary ideal of R if and only if $(I :_R s)$ is a primary ideal of R for some $s \in S$ if and only if $S^{-1}I$ is a primary ideal of $S^{-1}R$ and $S^{-1}I \cap R = (I :_R s)$ for some $s \in S$. Let R be a ring and let S be a multiplicatively closed subset of R. Recall from [13] that an increasing sequence $(I_k)_{k \in \mathbb{N}}$ of ideals of R is called S-*stationary* if there exist a positive integer n and an $s \in S$ such that for each $k \geq n$, $sI_k \subseteq I_n$. In [20], the author linked the S-primary property with the S-stationary notion. He proved that if I is an S-primary ideal of R, then the ascending sequence of ideal $(I :_R r) \subseteq (I :_R r^2) \subseteq \cdots$ is S-stationary for all $r \in R$. He also, gave an example to show that the reverse of this implication is not true in general.

Motivated by the research work presented in [20] on S-primary ideals and by the fact that the quasi-primary ideals also play an important role in the theory of commutative rings (see also [10]), in this article, we introduce the concept of quasi-S-primary ideals of a commutative ring and study its basic properties. We say a proper ideal I of R disjoint from S a *quasi-S-primary* ideal if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. This fixed element $s \in S$ is called a *quasi-S-primary element* of I. In the first part of this article, we prove some basic properties of quasi-S-primary ideals. We give an analogue results of the prime avoidance lemma. We show that if S is a multiplicatively closed subset of a ring R and I an ideal of R with $I \subseteq \bigcup_{i=1}^{n} I_j$ where $I_1, \ldots, I_n$ are quasi-S-primary ideals of R, then there exists $s \in S$ and $k \in \{1, \ldots, n\}$ such that $sI \subseteq \sqrt{I_k}$. Let P be an ideal of

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R disjoint with S. According to [14], P is called an *S-prime* ideal of R if there exists an $s \in S$ such that for all $a, b \in R$ if $ab \in P$, then $sa \in P$ or $sb \in P$. It is clear that every S-prime ideal of R is a quasi-S-prime ideal of R. In Theorem 2, we give a necessary and sufficient condition for a quasi-S-primary ideal of R to be S-prime. We show that the following statements are equivalent: (1) I is an S-prime ideal of R; (2) I is an S-primary ideal of R and there exists an $s \in S$ such that for each $x \in R$, $sx^2 \in I$ implies $sx \in I$; (3) I is a quasi-S-primary ideal of R and there exists an $s \in S$ such that for each $x \in R$, $sx^2 \in I$ implies $sx \in I$.

In Section 3, we investigate quasi-S-primary ideals of the polynomial ring $R[X]$, the power series ring $R[\![X]\!]$, the Serre's conjecture ring $R[X]_U$ and the Nagata ring $R[X]_N$ (the concepts of the Serre's conjecture ring and the Nagata ring will be reviewed in Section 3). Let I be an ideal of R disjoint with S. We show that I is a quasi-S-primary ideal of R if and only if $I[X]_N$ is a quasi-S-primary ideal of $R[X]_N$ if and only if $I[X]_U$ is a quasi-S-primary ideal of $R[X]_U$. For the power series ring $R[\![X]\!]$, we prove that if R is an S-Noetherian ring (a commutative ring R is called *S-Noetherian* if for each ideal I of R, $sI \subseteq J \subseteq I$ for some finitely generated ideal J of R and some $s \in S$.) and I an ideal of R disjoint with S, then I is a quasi-S-primary ideal of R if and only if $I[\![X]\!]$ is quasi-S-primary in $R[\![X]\!]$.

Finally, in Section 4, we give a relationship between quasi-S-primary ideals of a ring R and those of the amalgamated rings $R \bowtie^f J$. First let us recall the following notions. Let R and R' be two rings, $f : R \to R'$ be a homomorphism and J be an ideal of R'. The amalgamation of R and R' along J with respect to f is the subring $R \bowtie^f J = \{(a, f(a) + j) : a \in R, j \in J\}$, of $R \times R'$ introduced by D'Anna et. al. in [5]. Let I and K be ideals of the rings R and $f(R) + J$, respectively. In [8], two types of ideals of $R \bowtie^f J$ are studied: $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\overline{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in K\}$. For more detail regarding to amalgamated rings, the reader may refer to [6–8]. Let S be a multiplicatively closed subset of R. It is easily to prove that $S \bowtie^f J = \{(s, f(s) + j) : s \in S, j \in J\}$ and $W = \{(s, f(s)) : s \in S\}$ are multiplicatively closed subsets of $R \bowtie^f J$. Consider the amalgamation of rings R and R' along the ideal J of R' with respect to a homomorphism f. For an ideal I of R disjoint with S, we show that the following statements are equivalent.

(1) $I \bowtie^f J$ is a quasi-W-primary ideal of $R \bowtie^f J$.

(2) $I \bowtie^f J$ is a quasi-$(S \bowtie^f J)$-primary ideal of $R \bowtie^f J$.

(3) I is a quasi-S-primary ideal of R.

Let T be a multiplicatively closed subset of the ring R'. Then one can easily verify that the set $\bar{T}^f = \{(s, f(s) + j) : s \in R, j \in J, f(s) + j \in T\}$ is a multiplicatively closed subset of $R \bowtie^f J$. We end this paper by the following result: for an ideal K of R' and a multiplicatively closed subset T of R' disjoint with K, $\bar{K}^f$ is a quasi-$\bar{T}^f$-primary ideal of $R \bowtie^f J$ if and only if K is a quasi-T-primary ideal of R'.

2. Basic results

We start this section by introducing the notion of quasi-S-primary ideals.

Definition 1. Let S be a multiplicatively closed subset of a ring R. An ideal I of R disjoint with S is said to be a quasi-S-primary ideal if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$, then $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. This fixed element $s \in S$ is called a quasi-S-primary element of I.

Remark 1. Let R be a ring and S a multiplicatively closed subset of R.

(1) Every S-primary ideal of R is a quasi-S-primary ideal and clearly the two concepts coincide for radical ideals.

(2) For a proper ideal I of R, I is a quasi-S-primary ideal if and only if $\sqrt{I}$ is an S-prime (S-primary) ideal.

(3) Let $T \subseteq S$ be two multiplicatively closed subsets of R. If I is a quasi-T-primary ideal such that $I \cap S = \emptyset$, then I is also a quasi-S-primary ideal. Conversely, if for each $s \in S$, there is an element $t \in T$ such that $st \in T$ and I is a quasi-S-primary ideal of R, then I is a quasi-T-primary ideal of R. In particular, a quasi-primary ideal disjoint with S is a quasi-S-primary ideal. Additionally, if $S \subseteq U(R)$, then the concepts of quasi-primary ideals and quasi-S-primary ideals coincide. (see also Proposition 5).

We give the following examples to show that the converses of these implications in Remark above are not true in general.

Example 1. Consider the multiplicatively closed subset $S = \{\overline{1}, \overline{3}, \overline{9}\}$ and the ideal $I = \{\overline{0}, \overline{12}\}$ of the ring $\mathbb{Z}_{24}$. Then I is not a quasi-primary ideal since $\overline{3} \cdot \overline{4} \in I$ but neither $\overline{3} \in \sqrt{I} = (\overline{6})$ nor $\overline{4} \in \sqrt{I}$. However, it is a quasi-S-primary ideal with a quasi-S-primary element $s = \overline{3}$.

Example 2. Let $R = \mathbb{Z}[X]$, $P = 4X\mathbb{Z}[X]$ and $S = \{4^n \mid n \in \mathbb{N} \cup \{0\}\}$. Then P is an S-prime ideal of R; so a quasi-S-primary ideal. Note that P is not a quasi-primary ideal because $4X \in P$ and for each $n \in \mathbb{N}$, $4^n \notin P$ and $X^n \notin P$. Thus P is a quasi-S-primary ideal of R which is not a quasi-primary ideal.

Example 3. Let p and q be distinct prime integers. Consider the ring of polynomials $R = \mathbb{Z}+px\mathbb{Z}[x]/x^{n+1}\mathbb{Z}[x]$ and the multiplicatively closed subset $S = \{q^n : n \geq 0\}$ of R. Then the ideal $J = (p^2x^2, px^3, x^4, x^5, x^6)$ is a quasi-S-primary ideal as $\sqrt{J} = (px, x^2, x^3)$ is prime (so it is S-prime). However, J is not S-primary since $p^2x^2 \in J$ but neither $sp^2 \in \sqrt{J}$ nor $sx^2 \in J$ for each $s \in S$.

The saturation of S is the set $S^* = \{r \in R : \frac{r}{1}$ is a unit in $S^{-1}R\}$ or equivalently, $S^* = \{x \in R : xy \in S$ for some $y \in R\}$ is a multiplicatively closed subset of R and $S \subseteq S^*$, see [12]. Our next theorem gives several equivalent conditions for an ideal I disjoint with S to be quasi-S-primary.

Theorem 1. *Let I be an ideal of R disjoint with S. Then the following assertions are equivalent.*

(1) I is a quasi-S-primary ideal of R.

(2) I is a quasi-S^*-primary ideal of R.

(3) There exists an $s \in S$ such that $(I : a) \subseteq (\sqrt{I} : s)$ for each $a \notin (\sqrt{I} : s)$.

(4) There exists an $s \in S$ such that for any $a \in R$ and any ideal K of R, if $aK \subseteq I$, then $sa \subseteq \sqrt{I}$ or $sK \subseteq \sqrt{I}$.

(5) There exists an $s \in S$ such that for any two ideals J, K of R, if $JK \subseteq I$, then $sJ \subseteq \sqrt{I}$ or $sK \subseteq \sqrt{I}$.

(6) There exists an $s \in S$ such that for all ideals $I_1, \ldots, I_n$ of R, if $I_1 \cdots I_n \subseteq I$, then $sI_k \subseteq \sqrt{I}$ for some $k \in \{1, \ldots, n\}$.

(7) There exists an $s \in S$ such that for all $a_1, \ldots, a_n \in R$, if $a_1 \cdots a_n \subseteq I$, then $sa_k \subseteq \sqrt{I}$ for some $k \in \{1, \ldots, n\}$.

Proof. (1)⇒(2). Since $S \subseteq S^*$, it is sufficient to show that $S^* \cap I = \emptyset$ by Remark 1(3). If $\alpha \in S^* \cap I$, then there exists an $s \in S$ such that s is a multiple of α; so $s = \alpha r$ for some $r \in R$. Thus, we conclude $s \in S \cap I$, a contradiction.

(2)⇒(3). Let s^* be a quasi-S^*-primary element of I. There exists an $s \in S$ such that $s = s^*r$ for some $r \in R$. Let $a \in R \backslash (\sqrt{I} : s)$. We show that $(I : a) \subseteq (\sqrt{I} : s)$. Let b be an element of $(I : a)$. Since I is a quasi-S^*-primary ideal of R, $s^*a \in \sqrt{I}$ or $s^*b \in \sqrt{I}$; so $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. This implies that $sb \in \sqrt{I}$ because $sa \notin \sqrt{I}$. Thus $b \in (\sqrt{I} : s)$, as needed.

(3)⇒(4). Take the fixed s in (3) and assume that $aK \subseteq I$ and $sa \notin \sqrt{I}$. Then $a \notin (\sqrt{I} : s)$ and from (3), we get $K \subseteq (I : a) \subseteq (\sqrt{I} : s)$, and thus $sK \subseteq \sqrt{I}$.

(4)⇒(5). Take the fixed s in (4) and suppose that $JK \subseteq I$ and $sJ \nsubseteq \sqrt{I}$ for some ideals J, K of R. Then there exists $a \in J$ with $sa \notin \sqrt{I}$. Since $aK \subseteq I$ and $sa \notin \sqrt{I}$, we conclude from (4) that $sK \subseteq \sqrt{I}$, we are done.

(5)⇒(1). Let $a, b \in R$ with $ab \in I$. We conclude the result by taking $J =< a >$ and $K =< b >$ in (5).

(1)⇔(6). Suppose that I is a quasi-S-primary ideal of R. Then $\sqrt{I}$ is an S-prime ideal by Remark 1(2) and the claim follows from [14, Corollary 1]. The converse part is clear by putting $n = 2$ and (5)⇒(1).

(6)⇒(7). Suppose $a_1 \cdots a_n \subseteq I$ for some $a_1, \ldots, a_n \in R$. Put $I_j =< a_j >$ for each $j \in \{1, \ldots, n\}$. Hence, $sa_k \in sI_k \subseteq \sqrt{I}$ for some $k \in \{1, \ldots, n\}$ by our assumption (6).

(7)⇒(1). Take $n = 2$. □

Lemma 1. *Let I be an ideal of R disjoint with S. Then the following conditions are equivalent.*

1. *I is a quasi-S-primary ideal of R.*
2. *There exists an $s \in S$ such that $(\sqrt{I} : s)$ is a prime ideal of R.*

Proof. This follows from [14, Remark 1] and the fact that I is a quasi-S-primary ideal of R if and only if $\sqrt{I}$ is an S-prime ideal of R. □

Proposition 1. *Let I be an ideal of R with $I \subseteq \bigcup_{i=1}^{n} I_j$ where $I_1, \ldots, I_n$ are quasi-S-primary ideals of R. Then there exists $s \in S$ and $k \in \{1, \ldots, n\}$ such that $sI \subseteq \sqrt{I_k}$.*

Proof. Suppose that $I_1, \ldots, I_n$ are quasi-S-primary ideals and $I \subseteq \bigcup_{i=1}^{n} I_j$. Then $I \subseteq \bigcup_{i=1}^{n} (\sqrt{I_j} : s_j)$ and $(\sqrt{I_j} : s_j)$ is a prime ideal for each $j \in \{1, \ldots, n\}$ by Lemma 1. Thus, the prime avoidance lemma implies that there exists $k \in \{1, \ldots, n\}$ such that $I \subseteq (\sqrt{I_k} : s_k)$, and therefore $s_k I \subseteq \sqrt{I_k}$. □

Next we give a necessary and sufficient condition for a quasi-S-primary ideal of R to be S-prime.

Theorem 2. *Let P be an ideal of R disjoint with S. Then the following conditions are equivalent.*

(1) P is an S-prime ideal of R.

(2) P is an S-primary ideal of R and there exists an $s \in S$ such that for each $x \in R$, $sx^2 \in P$ implies $sx \in P$.

(3) P is a quasi-S-primary ideal of R and there exists an $s \in S$ such that for each $x \in R$, $sx^2 \in P$ implies $sx \in P$.

Proof. The implications (1)⇒ (2) ⇒(3) are obvious.

(3)⇒(1). Assume that P is a quasi-S-primary ideal of R and there exists an $s \in S$ such that for each $x \in R$, $sx^2 \in P$ implies $sx \in P$. Then there exists a $t \in S$ such that for all $a, b \in R$ if $ab \in P$, then $ta \in \sqrt{P}$ or $tb \in \sqrt{P}$. Put $s' := st \in S$. Let $a, b \in R$ such that $ab \in P$. We show that either $s'a \in P$ or $s'b \in P$. If $ta \in \sqrt{P}$, then $(ta)^n \in P$ for some $n \in \mathbb{N}$. If n is even, then $n = 2k$ for some $k \in \mathbb{N}$; so by hypothesis, $s(ta)^k \in P$. If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{N}$, which implies that $(ta)^{2k+2} = (ta)^{n+1} \in P$; so by hypothesis, $s(ta)^{k+1} \in P$. Thus there exists an $n_1 < n$, ($n_1 = k$ if n is even and $n_1 = k + 1$ if n is odd) such that $s(ta)^{n_1} \in P$. We continue this process, we obtain, $s(ta)^2 \in P$ or $s(ta)^3 \in P$. By using the hypothesis, we get $s'a = sta \in P$. In the same way we can prove that if $tb \in \sqrt{P}$, then $s'b \in P$. Hence, P is an S-prime ideal of R. □

In the particular case when S consists of units of R, we obtain the following result.

Corollary 1. *Let P be a proper ideal of a ring R. Then P is a prime ideal of R if and only if P is a quasi-primary ideal of R and for each $x \in R$, $x^2 \in P$ implies $x \in P$.*

Proposition 2. *Let I be a quasi-S-primary ideal of R. For an ideal J of R with $J \cap S \neq \emptyset$, $I \cap J$ and IJ are quasi-S-primary ideals of R.*

Proof. Since $I \cap S = \emptyset$, clearly we have $(I \cap J) \cap S = \emptyset$ and $IJ \cap S = \emptyset$. Suppose that $s \in S$ is a quasi-S-primary element of I. Let $ab \in I \cap J$. Then $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. Choose $t \in J \cap S$. Then $sta \in \sqrt{I} \cap \sqrt{J} = \sqrt{I \cap J}$ or $stb \in \sqrt{I \cap J}$. Thus $I \cap J$ is a quasi-S-primary ideal of R with a quasi-S-primary element $s' = st$. The proof is similar for IJ. ☐

It is clear that the intersection of quasi-S-primary ideals of a commutative ring is not a quasi-S-primary ideal in general. Our next proposition shows that under an additional assumption the intersection of quasi-S-primary ideals is a quasi-S-primary ideal.

Proposition 3. *If $I_1, \ldots, I_n$ are quasi-S-primary ideals of R with the same radical, then $\bigcap_{i=1}^{n} I_i$ is a quasi-S-primary ideal of R.*

Proof. Put $\mathfrak{q} := \bigcap_{i=1}^{n} I_i$. Since the ideals $I_1, \ldots, I_n$ are disjoint with S, we obtain $S \cap \mathfrak{q} = \emptyset$. Let $1 \leq i \leq n$. Since I_i is a quasi-S-primary ideal of R, there exists an $s_i \in S$ such that for each $a, b \in R$ with $ab \in I_i$, we have either $s_i a \in \sqrt{I_i}$ or $s_i b \in \sqrt{I_i}$. Put $s := s_1 s_2 \cdots s_n$. Then $s \in S$. Let $a, b \in R$ such that $ab \in \mathfrak{q}$. Assume that $sa \notin \sqrt{\mathfrak{q}}$. Then there exists a $1 \leq j \leq n$ such that $s_j a \notin \sqrt{I_j}$. Since I_j is a quasi-S-primary ideal of R (with a quasi-S-primary element s_j) and $ab \in I_j$, we get $s_j b \in \sqrt{I_j}$. This implies that $sb \in \sqrt{I_j}$. By hypothesis, for each $1 \leq i \leq n$, $\sqrt{I_i} = \sqrt{I_j}$; so $sb \in \bigcap_{i=1}^{n} \sqrt{I_i} = \sqrt{\mathfrak{q}}$. Thus $\mathfrak{q}$ is a quasi-S-primary ideal of R. ☐

Recall from [13] that an increasing sequence $(I_k)_{k \in \mathbb{N}}$ of ideals of a ring R is called *S-stationary* if there exist a positive integer n and an $s \in S$ such that for each $k \geq n$, $sI_k \subseteq I_n$. We next give a relationship between the "S-stationary" and "quasi-S-primary" notions. Note that the proof is inspired by a result proved by S. Visweswaran in [20].

Theorem 3. *Let I be an ideal of R disjoint with S. Consider the following statements:*

(1) I is a quasi-S-primary ideal of R.

(2) The ascending sequence $(\sqrt{I} : sr) \subseteq (\sqrt{I} : sr^2) \subseteq \cdots$ of ideals is stationary for some $s \in S$ and for all $r \in R$.

(3) The ascending sequence $(\sqrt{I} : r) \subseteq (\sqrt{I} : r^2) \subseteq \cdots$ of ideals is S-stationary for all $r \in R$.

Then (1)⇒ (2) ⇒(3).

Proof. (1)⇒ (2). Suppose that I is a quasi-S-primary ideal of R. There exists an $s \in S$ such that for each $a, b \in R$ with $ab \in I$, we have either $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. Let r be an element of R. We show that the sequence $(\sqrt{I} : sr) \subseteq (\sqrt{I} : sr^2) \subseteq \cdots$ is stationary.

First case: $r \notin (\sqrt{I} : s)$. We show that for each $n \in \mathbb{N}$, $(\sqrt{I} : sr^n) = (\sqrt{I} : s)$, which implies that the sequence $(\sqrt{I} : sr) \subseteq (\sqrt{I} : sr^2) \subseteq \cdots$ is stationary. Let $n \in \mathbb{N}$. It is clear that $(\sqrt{I} : s) \subseteq (\sqrt{I} : sr^n)$. Conversely, let $\alpha \in (\sqrt{I} : sr^n)$. Then $(s\alpha r)^k \in I$ for some $k \in \mathbb{N}$. Since I is a quasi-S-primary ideal of R with a quasi-S-primary element s, we get either $s\alpha^k \in \sqrt{I}$ or $s^{k+1}r^k \in \sqrt{I}$, which implies that $\alpha \in (\sqrt{I} : s)$ or $r \in (\sqrt{I} : s)$. Thus $\alpha \in (\sqrt{I} : s)$ since $r \notin (\sqrt{I} : s)$. Hence for each $n \in \mathbb{N}$, $(\sqrt{I} : sr^n) = (\sqrt{I} : s)$.

Second case: $r \in (\sqrt{I} : s)$. Then $sr \in \sqrt{I}$; so $(\sqrt{I} : sr^n) = R$, and hence the sequence $(\sqrt{I} : sr) \subseteq (\sqrt{I} : sr^2) \subseteq \cdots$ is stationary.

(2)⇒ (3). Let $r \in R$. By hypothesis, there exists an $n \in \mathbb{N}$ such that for each $k \geq n$, $(\sqrt{I} : sr^k) = (\sqrt{I} : sr^n)$ for all $n \in \mathbb{N}$. This implies that for each $k \geq n$, $s(\sqrt{I} : r^k) \subseteq (\sqrt{I} : r^n)$. Indeed, let $\alpha \in (\sqrt{I} : r^k)$. Then $s\alpha r^k \in \sqrt{I}$, which implies that $\alpha \in (\sqrt{I} : sr^k) = (\sqrt{I} : sr^n)$. This implies that $s\alpha r^n \in \sqrt{I}$, and thus $s\alpha \in (\sqrt{I} : r^n)$. Hence, the sequence $(\sqrt{I} : r) \subseteq (\sqrt{I} : r^2) \subseteq \cdots$ is S-stationary. ☐

Recall from [16] that a ring R is said to *satisfy accr* (respectively, *satisfy accr**) if the ascending sequence of ideals $(I :_R B) \subseteq (I :_R B^2) \subseteq (I :_R B^3) \subseteq \cdots$ is stationary for any ideal I of R and for any finitely generated (respectively, principal) ideal B of R. It was shown in [16] that the properties accr and accr* are equivalent. Our next example shows that the implication (3)⇒ (1) of Theorem 3 may fail to hold.

Example 4. Let R be a ring satisfying accr, $S \subseteq U(R)$ and I a non-quasi-primary ideal of R (for example, $R = \mathbb{Z}_{24}$ and $I = \{\overline{0}, \overline{12}\}$ as in Example 1). Since R satisfies accr, the ascending sequence of ideals $(\sqrt{I} : r) \subseteq (\sqrt{I} : r^2) \subseteq \cdots$ is stationary for all $r \in R$. But I is not a quasi-primary ideal of R.

Proposition 4. *Let $f : R \to R'$ be a ring homomorphism and S be a multiplicatively closed subset of R such that $f(S)$ does not contain zero. Then the following statements hold.*

(1) *If f is an epimorphism and I is a quasi-S-primary ideal of R containing $Ker(f)$, then $f(I)$ is a quasi-$f(S)$-primary ideal of R'.*

(2) *If J is a quasi-$f(S)$-primary ideal of R', then $f^{-1}(J)$ is a quasi-S-primary ideal of R.*

Proof. (1) First, we show that $f(I) \cap f(S) = \emptyset$. Assume not. Then there exists $u \in f(I) \cap f(S)$, which implies $u = f(x) = f(s)$ for some $x \in I$ and $s \in S$. Hence, $x - s \in Ker(f) \subseteq I$ and $s \in I \cap S$, which is a contradiction. Suppose $a'b' \in f(I)$ for some $a, b \in R'$. Say $a' := f(a)$ and $b' := f(b)$ for some $a, b \in R$. This yields that there exists an $s \in S$ such that $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. Thus, clearly we have $f(s)a' \in f(\sqrt{I})$ or $f(s)b' \in f(\sqrt{I})$, we are done as $f(\sqrt{I}) = \sqrt{f(I)}$.

(2) If $s \in f^{-1}(J) \cap S$, then $f(s) \in J \cap S$, which is a contradiction. Hence, $f^{-1}(J) \cap S = \emptyset$. Suppose $ab \in f^{-1}(J)$ for some $a, b \in R$. Then $f(ab) = f(a)f(b) \in J$ and since J is a quasi-$f(S)$-primary ideal of R', there exists $f(s) \in f(S)$ such that $f(s)f(a) \in \sqrt{J}$ or $f(s)f(b) \in \sqrt{J}$. Thus, $sa \in f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}$ or $sb \in f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}$, as needed. ☐

Let I be an ideal of R disjoint with S. If we denote $r + I \in R/I$ by $\overline{r}$, then clearly the set $\overline{S} = \{\overline{s} : s \in S\}$ is a multiplicatively closed subset of R/I. In view of Proposition 4, we conclude the following result for quasi-$\overline{S}$-primary ideals of R/I.

Corollary 2. *Let $I \subseteq J$ be two ideals of R such that $S \cap J = \emptyset$. Then the following statements hold.*

(1) If J is a quasi-S-primary ideal of R, then J/I is a quasi-$\overline{S}$-primary ideal of R/I.

(2) If R is a subring of R' and I' is a quasi-S-primary ideal of R', then $I' \cap R$ is a quasi-S-primary ideal of R.

Proof. (1) First, observe that $(J/I) \cap \overline{S} = \emptyset$ if and only if $J \cap S = \emptyset$. Taking the canonical epimorphism $\pi : R \to R/I$ in Proposition 4 (1), we are done.

(2) It is clear that $(I' \cap R) \cap S = \emptyset$. The claim follows from Proposition 4 (2) considering the natural injection $i : R \to R'$. □

We have a condition under which a quasi-S-primary ideal needs to be quasi-primary. By $Z(R)$, we denote the set of zero divisors of a ring R.

Proposition 5. *Let I be an ideal of R disjoint from S. Assume that $\overline{S} \cap Z(R/\sqrt{I}) = \emptyset$. Then I is quasi-S-primary if and only if I is quasi-primary.*

Proof. Let $a \in (\sqrt{I} : s)$. Then $sa \in \sqrt{I}$, and hence $(s + \sqrt{I})(a + \sqrt{I}) = \sqrt{I}$. Since $\overline{S} \cap Z(R/\sqrt{I}) = \emptyset$, we conclude $a + \sqrt{I} = \sqrt{I}$; i.e., $a \in \sqrt{I}$. Thus $(\sqrt{I} : s) \subseteq \sqrt{I}$ and we get the equality as the converse inclusion always holds. From Lemma 1, $\sqrt{I} = (\sqrt{I} : s)$ is prime, so we get the claim. The converse part follows by Remark 1(2). □

Proposition 6. *Let $S \subseteq reg(R)$ be multiplicatively closed subset of a ring R and I an ideal of R disjoint with S. Then I is a quasi-S-primary ideal of R if and only if $S^{-1}I$ is a quasi-primary ideal of $S^{-1}R$ and $(S^{-1}\sqrt{I}) \cap R = (\sqrt{I} : s)$ for some $s \in S$.*

Proof. Let $s \in S$ be a quasi-S-primary element of I. Notice that $S^{-1}I \cap S^{-1}R = \emptyset$ as $S \cap I = \emptyset$. Suppose $a, b \in R$ and $s_1, s_2 \in S$ with $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$ and $\frac{a}{s_1} \notin \sqrt{S^{-1}I}$. Then $tab \in I$ for some $t \in S$ and $sa \notin \sqrt{I}$. Since I is a quasi-S-primary ideal, we conclude $stb \in \sqrt{I}$. Thus, $\frac{b}{t_2} = \frac{stb}{stt_2} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$ and $S^{-1}I$ is a quasi-primary ideal of $S^{-1}R$. Let $x \in (S^{-1}\sqrt{I}) \cap R$. Then $x = \frac{a}{u}$ for some $a \in \sqrt{I}$ and $u \in S$, which follows $xu \in \sqrt{I}$. This yields that $sx \in \sqrt{I}$ or $su \in \sqrt{I}$. Since $S \cap \sqrt{I} = \emptyset$, the latter case is impossible. Thus $sx \in \sqrt{I}$, and hence $x \in (\sqrt{I} : s)$. Therefore, $(S^{-1}\sqrt{I}) \cap R \subseteq (\sqrt{I} : s)$. For the converse inclusion, let $y \in (\sqrt{I} : s)$. Then $sy \in \sqrt{I}$. As $S \subseteq reg(R)$, $f : R \to S^{-1}R$ defined by $f(a) = \frac{a}{1}$ is an injective ring homomorphism and $y = \frac{y}{1} = \frac{sy}{s} \in S^{-1}\sqrt{I}$. Thus, $y \in (S^{-1}\sqrt{I}) \cap R$ and we have the equality.

Conversely, suppose that $ab \in I$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}I$, which implies $\left(\frac{a}{1}\right)^n \in S^{-1}I$ for some positive integer n or $\left(\frac{b}{1}\right)^m \in S^{-1}I$ for some positive integer m. If the former case holds, $va^n \in I$ for some $v \in S$, which implies $va \in \sqrt{I}$. Hence, $a = \frac{a}{1} = \frac{va}{v} \in (S^{-1}\sqrt{I}) \cap R = (\sqrt{I} : s)$ for some $s \in S$. Thus $sa \in \sqrt{I}$. For the case of $\left(\frac{b}{1}\right)^m \in S^{-1}I$, similarly we get $sb \in \sqrt{I}$. Therefore, I is a quasi-S-primary ideal of R. □

Next, we characterize quasi-S-primary ideals in a cartesian product of rings.

Theorem 4. *Let $R = R_1 \times R_2$ and $S = S_1 \times S_2$ where S_1, S_2 are multiplicatively closed subsets and I_1, I_2 be ideals of rings R_1, R_2, respectively. Then $I = I_1 \times I_2$ is a quasi-S-primary ideal of R if and only if either I_1 is a quasi-S_1-primary ideal of R_1 and $S_2 \cap I_2 \neq \emptyset$ or I_2 is a quasi-S_2-primary ideal of R_2 and $S_1 \cap I_1 \neq \emptyset$.*

Proof. Let $s = (s_1, s_2)$ be a quasi-S-primary element of I. Assume on the contrary that $S_1 \cap I_1 = S_2 \cap I_2 = \emptyset$. Let $(a, b) \in I$. Then $(a, 1)(1, b) \in I$, which yields either $(s_1, s_2)(a, 1) \in \sqrt{I}$ or $(s_1, s_2)(1, b) \in \sqrt{I}$. Thus, we get either $s_2^n \in S_2 \cap I_2$ for some positive integer n or $s_1^m \in S_1 \cap I_1$ for some positive integer m, a contradiction. Without loss of generality, we may assume that $S_1 \cap I_1 \neq \emptyset$ and we will prove that I_2 is a quasi-S_2-primary ideal of R_2. First, $S_2 \cap I_2 = \emptyset$ as $S \cap I = \emptyset$. Suppose $ab \in I_2$ for some $a, b \in R_2$. Choose $t \in S_1 \cap I_1$. Hence $(t, a)(1, b) \in I$ implies either $s(t, a) \in \sqrt{I}$ or $s(1, b) \in \sqrt{I}$. Therefore, $s_2 a \in \sqrt{I_2}$ or $s_2 b \in \sqrt{I_2}$, as needed.

Conversely, we may assume without loss of generality that I_1 is a quasi-S_1-primary ideal of R_1 with a quasi-S_1-primary element $s_1 \in S_1$ and $S_2 \cap I_2 \neq \emptyset$. Choose $s_2 \in S_2 \cap I_2$. Let $(a_1, a_2)(b_1, b_2) \in I$ for some $a_1, b_1 \in R_1$ and $a_2, b_2 \in R_2$. Hence $a_1 b_1 \in I_1$, which implies $s_1 a_1 \in \sqrt{I_1}$ or $s_1 b_1 \in \sqrt{I_1}$. Now set $s = (s_1, s_2) \in S$. Observe that $s(a_1, a_2) \in \sqrt{I}$ or $s(b_1, b_2) \in \sqrt{I}$ and thus I is a quasi-S-primary ideal of R. □

The following example shows that the condition $S_1 \cap I_1 \neq \emptyset$ $(S_2 \cap I_2 \neq \emptyset)$ is crucial. Note that $I = I_1 \times I_2$ needs not to be a quasi-$S = S_1 \times S_2$-primary even if both of I_1 and I_2 are S_i-primary for $i = 1, 2$, respectively.

Example 5. Let p and q be distinct prime integers. Consider the multiplicatively closed subsets $S = \{p^n : n \geq 0\}$ and $T = \{q^n : n \geq 0\}$ of the ring $\mathbb{Z}$. Then $I = q\mathbb{Z}$ and $J = p\mathbb{Z}$ are clearly S-primary and T-primary ideals of $\mathbb{Z}$, respectively, but $I \times J$ is not a quasi-$S \times T$-primary ideal of $\mathbb{Z} \times \mathbb{Z}$. Indeed, $(1, p)(q, 1) \in I \times J$ but neither $(s, t)(1, p) \in \sqrt{I \times J} = q\mathbb{Z} \times p\mathbb{Z}$ nor $(s, t)(q, 1) \in \sqrt{I \times J}$ for each $(s, t) \in S \times T$.

In view of Theorem 4, we conclude the following result.

Corollary 3. *Let $R = R_1 \times \cdots \times R_n$ and $S = S_1 \times \cdots \times S_n$ where S_i's are multiplicatively closed subsets of R_i's for all $i \in \{1, \ldots, n\}$, respectively. Then $I = I_1 \times \cdots \times I_n$ is a quasi-S-primary ideal of R if and only if I_k is a quasi-S_k-primary ideal of R_k for some $k \in \{1, \ldots, n\}$ and $S_j \cap I_j \neq \emptyset$ for all $j \in \{1, \ldots, n\} \backslash \{k\}$.*

Recall that a proper ideal I of a commutative ring R is said to be weakly irreducible provided that for each pair of ideals A and B of R, $A \cap B \subseteq I$ implies that either $A \subseteq \sqrt{I}$ or $B \subseteq \sqrt{I}$ [18]. In the following proposition, we show the relationship between weakly irreducible and quasi-S-primary ideals. For a proper ideal J of R, the notation $Z_J(R)$ denotes $\{r \in R : ra \in I$ for some $a \in R \backslash J\}$.

Proposition 7. *Let I be a quasi-S-primary ideal of R satisfying $Z_{\sqrt{I}}(R) \cap S = \emptyset$. Then I is weakly irreducible.*

Proof. Let J, K be ideals of R with $J \cap K \subseteq I$. Then $JK \subseteq I$ which implies that there exists an $s \in S$ such that $sJ \subseteq \sqrt{I}$ or $sK \subseteq \sqrt{I}$ by Theorem 1(5). Suppose that $sJ \subseteq \sqrt{I}$. If there is a $j \in J \backslash \sqrt{I}$, then $s \in Z_{\sqrt{I}}(R)$ which contradicts to our assumption. Thus, $J \subseteq \sqrt{I}$. Similarly, if $sK \subseteq \sqrt{I}$, then we conclude $K \subseteq \sqrt{I}$, which shows that I is weakly irreducible. □

In the ring $\mathbb{Z}_n$, we have the multiplicatively closed subset $\left\{1, \bar{p}, \bar{p}^2, \bar{p}^3, \ldots\right\}$ of $\mathbb{Z}_n$, denoted by S_p, for any prime p dividing n. In the following, we characterize all S_p-quasi-primary ideals of $\mathbb{Z}_n$.

Theorem 5. *Let $n \in \mathbb{N}$, p be a prime integer dividing n and I be a proper ideal of the ring $\mathbb{Z}_n$. Then I is an S_p-quasi-primary ideal of $\mathbb{Z}_n$ if and only if $I =< q^k >$ or $I =< p^t q^k >$ where q is a prime divisor of n distinct from p and $t, k \geq 1$.*

Proof. For a positive integer n, we define $\omega(n)$ to be the unique number of unique prime factors of n, for example $\omega(24) = 2$. To avoid the trivial case, we assume that $\omega(n) \geq 2$.

Suppose that $I =< m >$. It is clear that $\omega(m) \leq \omega(n)$. Let I be an S_p-quasi-primary ideal of $\mathbb{Z}_n$ where p divides n. We have the following cases.

Case I. Suppose that $\omega(n) = 2$ and say p and q are prime factors of n. If $\omega(m) = 1$, then $I =< q^k >$ as $I \cap S_p = \emptyset$ for some positive integer k. Suppose that $\omega(m) = 2$. Then $I =< p^t q^k >$ where t and k are positive integers.

Case II. Suppose that $\omega(n) \geq 3$. If $\omega(m) = 1$, then we are done similar to the first case. Let $\omega(m) = 2$ and say p_1 and p_2 are factors of m. Assume that p is distinct from p_1 and p_2. Hence, $p_1^k p_2^k \in I$ for some positive integer k but neither $sp_1^k \in \sqrt{I} =< p_1 p_2 >$ nor $sp_2^k \in \sqrt{I}$ for all $s \in S_p$, we get a contradiction. Thus $p_1 = p$ or $p_2 = p$.

Next, we show that the case of $\omega(m) \geq 3$ is impossible. Assume that $\omega(m) = 3$ and say, p_1, p_2, p_3 are prime factors of m. Without loss of generality, we show that I is not an S_{p_1}-quasi-primary ideal. Indeed, $p_1^k p_2^k p_3^k \in I$ for some positive integer k but neither $s(p_1^k p_2^k) \in \sqrt{I} =< p_1 p_2 p_3 >$ nor $sp_3^k \in \sqrt{I}$. It can be easily seen that I is not an S_p-quasi-primary ideal of $\mathbb{Z}_n$ for all prime factors p of n distinct from p_1, p_2, and p_3. The general case of $\omega(m) \geq 3$ can be verified by using the similar manner above.

Conversely, if $I =< q^k >$ where $q \neq p$ divides n, then I is a primary ideal with $I \cap S_p = \emptyset$; and so I is an S_p-quasi-primary ideal of $\mathbb{Z}_n$. Now, let $I =< p^t q^k >$ and suppose that $ab \in I$ for some $a, b \in \mathbb{Z}_n$. If q divides a, then $sa \in \sqrt{I}$ for all $s \in S_p$. If q does not divide a, then clearly q divides b which implies $sb \in \sqrt{I}$ for all $s \in S_p$, we are done. □

We present the following result which enables us to built some interesting examples for quasi-S-primary ideals that are not quasi-primary.

Proposition 8. *Let I be an ideal of R disjoint with S. Let $s \in S$ such that $\bigcap_{n\geq 1}\sqrt{s^nR} = (0)$. Then either $I = (0)$ or sI is not a quasi-primary ideal of R.*

Proof. Assume that $I \neq (0)$. First, we show that $\sqrt{I} \neq \sqrt{sI}$. If $\sqrt{I} = \sqrt{sI}$, then $\sqrt{I} \subseteq \sqrt{s\sqrt{sI}} \subseteq \sqrt{s^2 I}$. By induction, for each $n \in \mathbb{N}$, $\sqrt{I} \subseteq \sqrt{s^n I}$. This implies that

$$\sqrt{I} \subseteq \bigcap_{n\geq 1}\sqrt{s^n I} \subseteq \bigcap_{n\geq 1}\sqrt{s^n R} = (0),$$

a contradiction since $I \neq (0)$.

Now, suppose that sI is a quasi-primary ideal of R, and let $\alpha \in \sqrt{I}\backslash\sqrt{sI}$. Since $s\alpha \in s\sqrt{I} \subseteq \sqrt{sI}$, we have either $s \in \sqrt{sI}$ or $\alpha \in \sqrt{sI}$. But $\alpha \notin \sqrt{sI}$ and $s \notin \sqrt{sI}$ since $I \cap S = \emptyset$; so we conclude that sI is not a quasi-primary ideal of R. □

Example 6. Let $R = K[X, Y]$ be the polynomial ring in two variables X, Y over a field K, $M = XR + YR$ and $S = R\backslash M$. Then S is a multiplicatively closed subset of R. Let $n \geq 1$, and let $I_n = M^n$. We show that $J_n = (X-1)I_n$ is a quasi-S-primary ideal of R but not a quasi-primary ideal of R. According to [4, Proposition 4.2], for each $n \geq 1$, I_n is a primary ideal of R. Since $(X-1) \in S$ and $I_n \neq (0)$, J_n is a quasi-S-primary ideal of R. Now, it is clear that $\bigcap_{n\geq 1}\sqrt{(X-1)^n R} = (0)$. Since $I_n \neq (0)$, $I_n \cap S = \emptyset$, and $(X-1) \in S$, by Proposition 8, $J_n = (X-1)I_n$ is not a quasi primary ideal of R.

We end this section by the following remark.

Remark 2. For any multiplicatively closed subset S of R, it is easy to show that if $\mathfrak{q}$ is a quasi-S-primary ideal of R, then for each $s \in S$, $s\mathfrak{q}$ is also a quasi-S-primary ideal of R. Then we have the following chain of quasi-S-primary ideals of R

$$\cdots \subseteq s^n\mathfrak{q} \subseteq s^{n-1}\mathfrak{q} \subseteq \cdots \subseteq s\mathfrak{q} \subseteq \mathfrak{q}.$$

We define the height of $\mathfrak{q}$, written SQ-ht$(\mathfrak{q})$ to be the supremum of the lengths of all chains of quasi-S-primary ideals contained in $\mathfrak{q}$. If there exists an $s \in S$ such that for each $n \in \mathbb{N}$, $s^n\mathfrak{q} \subsetneq s^{n-1}\mathfrak{q}$, then SQ-ht$(\mathfrak{q}) = \infty$. For example, let X be an indeterminate over R. It is clear that (X) is an S-prime ideal of $R[X]$. Then (X) is a quasi-S-primary ideal of R. Since for all $s \in S$ and all $n \in \mathbb{N}$, $s^n\mathfrak{q} \subsetneq s^{n-1}\mathfrak{q}$, we get SQ-ht$(X) = \infty$.

3. Polynomial and power series rings

Let R be a commutative ring, S be a multiplicatively closed subset of R and X be an indeterminate. In this section, we investigate quasi-S-primary ideals of the polynomial ring $R[X]$, the power series ring $R[[X]]$, the Serre's conjecture ring $R[X]_U$ and the Nagata ring $R[X]_N$ (the concepts of the Serre's conjecture ring and the Nagata ring will be reviewed in the end of this section).

Theorem 6. *Let I be an ideal of R disjoint with S. Then the following assertions are equivalent.*

(1) *I is a quasi-S-primary ideal of R*

(2) *$I[X]$ is a quasi-S-primary ideal of $R[X]$.*

(3) *$I + XR[X]$ is a quasi-S-primary ideal of $R[X]$.*

(4) *$I + XR[\![X]\!]$ is a quasi-S-primary ideal of $R[\![X]\!]$.*

Proof. (1)$\Leftrightarrow$ (2). I is a quasi-S-primary ideal of R if and only if $\sqrt{I}$ is S-prime by Remark 1(2) if and only if $\sqrt{I}[X] = \sqrt{I[X]}$ is S-prime by [14, Example 4] if and only if $I[X]$ is quasi-S-primary in $R[X]$ again by Remark 1(2).

(1) $\Leftrightarrow$ (3). Assume that I is a quasi-S-primary ideal of R and let $s \in S$ be a quasi-S-primary element of I. Let $f, g \in R[X]$ such that $fg \in I + XR[X]$. Since $f(0)g(0) \in I$, it follows that $sf(0) \in \sqrt{I}$ or $sg(0) \in \sqrt{I}$; so $sf \in \sqrt{I} + XR[X]$ or $sg \in \sqrt{I} + XR[X]$. But $\sqrt{I} + XR[X] \subseteq \sqrt{I + XR[X]}$, and thus $sf \in \sqrt{I + XR[X]}$ or $sg \in \sqrt{I + XR[X]}$. Hence $I + XR[X]$ is a quasi-S-primary of $R[X]$. The reverse implication is obvious.

(1) $\Leftrightarrow$ (4). In the same way as (1) $\Leftrightarrow$ (3). ☐

In the next proposition we give with an additional assumption a necessary and sufficient condition for the ideal $I[\![X]\!]$ to be quasi-S-primary in $R[\![X]\!]$. First, we need to collect some necessary notions. Recall from [1] that an ideal I of R is called *S-finite* if $sI \subseteq J \subseteq I$ for some finitely generated ideal J of R and some $s \in S$. Also R is called *S-Noetherian* if each ideal of R is S-finite.

Lemma 2. *For a ring R, we have:*

(1) *If R is S-Noetherian, then for each ideal I of R, $s\sqrt{I}[\![X]\!] \subseteq \sqrt{I[\![X]\!]} \subseteq \sqrt{I}[\![X]\!]$ for some $s \in S$.*

(2) *P is an S-prime ideal of R if and only if $P[\![X]\!]$ is an S-prime ideal of $R[\![X]\!]$.*

Proof. (1). First we show that $\sqrt{I[\![X]\!]} \subseteq \sqrt{I}[\![X]\!]$. Note that this inclusion is always true and does not require the assumption that R is S-Noetherian. Let P be a prime ideal of R such that $I \subseteq P$. Then $P[\![X]\!]$ is a prime ideal of $R[\![X]\!]$ containing $I[\![X]\!]$; so $\sqrt{I[\![X]\!]} \subseteq P[\![X]\!]$. This implies that

$$\sqrt{I[\![X]\!]} \subseteq \bigcap_{P\in \mathrm{Spec}(R),\, I\subseteq P} P[\![X]\!] = \Big(\bigcap_{P\in \mathrm{Spec}(R),\, I\subseteq P} P\Big)[\![X]\!] = \sqrt{I}[\![X]\!].$$

Now assume that R is S-Noetherian. Since $\sqrt{I}$ is S-finite, $s\sqrt{I} \subseteq J \subseteq \sqrt{I}$ for some finitely generated ideal J of R and some $s \in S$. We show that $s\sqrt{I}[\![X]\!] \subseteq \sqrt{I[\![X]\!]}$. Put $J := (a_1, \ldots, a_n)$. For each $1 \leq i \leq n$, there exists an $m_i \in \mathbb{N}$ such that $a_i^{m_i} \in I \subseteq I[\![X]\!]$. This implies that for each $1 \leq i \leq n$, $a_i \in \sqrt{I[\![X]\!]}$. Let $f = \sum_{j=0}^{\infty} b_j X^j \in \sqrt{I}[\![X]\!]$. For each $j \geq 0$, $sb_j \in J$; so $sb_j = \sum_{i=1}^{n} \alpha_{ij} a_i$, where $\alpha_{ij} \in R$. Thus

$$sf = \sum_{j=0}^{\infty} (sb_j)X^j = \sum_{j=0}^{\infty} \Big(\sum_{i=1}^{n} \alpha_{ij} a_i\Big) X^j = \sum_{i=1}^{n} a_i \Big(\sum_{j=0}^{\infty} \alpha_{ij} X^j\Big) \in \sqrt{I[\![X]\!]}.$$

Hence $s\sqrt{I}[\![X]\!] \subseteq \sqrt{I[\![X]\!]}$.

(2). Suppose that P is an S-prime ideal of R. Then by [14, Remark 1], $(P : s)$ is a prime ideal of R for some $s \in S$. It follows that $(P :_R s)[\![X]\!]$ is a prime ideal of $R[\![X]\!]$ by [3, Theorem 4]. This implies that $(P[\![X]\!] :_{R[\![X]\!]} s) = (P :_R s)[\![X]\!]$ is a prime ideal of $R[\![X]\!]$. Thus $P[\![X]\!]$ is an S-prime ideal of $R[\![X]\!]$ again by Lemma [14, Remark 1]. The reverse implication is obvious. ☐

Proposition 9. *Let R be an S-Noetherian ring and I an ideal of R disjoint with S. Then I is a quasi-S-primary ideal of R if and only if $I[\![X]\!]$ is quasi-S-primary in $R[\![X]\!]$.*

Proof. Suppose that I is a quasi-S-primary ideal of R. Then $\sqrt{I}$ is an S-prime ideal of R; so by Lemma 2(2), $\sqrt{I}[\![X]\!]$ is an S-prime ideal of $R[\![X]\!]$. Thus there exists an $s \in S$ such that for each $f, g \in R[\![X]\!]$ with $fg \in \sqrt{I}[\![X]\!]$, we have $sf \in \sqrt{I}[\![X]\!]$ or $sg \in \sqrt{I}[\![X]\!]$.

On the other hand, by Lemma 2(1), there exists a $t \in S$ such that $t\sqrt{I}[\![X]\!] \subseteq \sqrt{I[\![X]\!]} \subseteq \sqrt{I}[\![X]\!]$. Now, we show that $I[\![X]\!]$ is quasi-S-primary in $R[\![X]\!]$ with a quasi-S-primary element st. Let $f, g \in R[\![X]\!]$ such that $fg \in I[\![X]\!]$. Since $fg \in \sqrt{I}[\![X]\!]$, $sf \in \sqrt{I}[\![X]\!]$ or $sg \in \sqrt{I}[\![X]\!]$. This implies that $stf \in t\sqrt{I}[\![X]\!] \subseteq \sqrt{I[\![X]\!]}$ or $stg \in t\sqrt{I}[\![X]\!] \subseteq \sqrt{I[\![X]\!]}$. Hence, we obtain $stf \in \sqrt{I[\![X]\!]}$ or $stg \in \sqrt{I[\![X]\!]}$.

Conversely, if $I[\![X]\!]$ is a quasi-S-primary ideal of $R[\![X]\!]$, then $I[\![X]\!] \cap R = I$ is a quasi-S-primary ideal of R. ☐

Let R be a commutative ring with identity and let $R[X]$ be the polynomial ring over R. Let U be the set of monic polynomials in $R[X]$. Then U is a multiplicatively closed subset of $R[X]$ and the quotient ring $R[X]_U$ is called the *Serre's conjecture ring* of R. For an element $f \in R[X]$, $c(f)$ denotes the content ideal of f, i.e., the ideal of R generated by the coefficients of f. Let $N = \{f \in R[X] \mid c(f) = R\}$. Then it was shown that $N = R[X] \setminus \bigcup_{M \in Max(R)} MR[X]$ and N is a saturated multiplicatively closed subset of $R[X]$ consisting of regular elements of $R[X]$ [17, pp. 17 and 18] (or [15, Proposition 2.1(1)]). The quotient ring $R[X]_N$ is called the *Nagata ring* of R. For more on the Nagata ring, the readers can refer to [15] and [17].

Theorem 7. *Let I be an ideal of R disjoint with S. Then the following statements are equivalent.*

(1) I is a quasi-S-primary ideal of R.

(2) $I[X]_N$ is a quasi-S-primary ideal of $R[X]_N$.

(3) $I[X]_U$ is a quasi-S-primary ideal of $R[X]_U$.

Proof. First, it is easy to prove that for each $s \in S$,
$(\sqrt{I} : s)[X]_N = (\sqrt{I}[X]_N :_{R[X]_N} s) = (\sqrt{I[X]_N} :_{R[X]_N} s)$.

(1) ⇒ (2). Assume that I is a quasi-S-primary ideal of R. Then by [14, Remark 1], there exists an $s \in S$ such that $(\sqrt{I} : s)$ is a prime ideal of R. This implies that $(\sqrt{I} : s)[X]$ is a prime ideal of the polynomial ring $R[X]$. We show that $(\sqrt{I} : s)[X] \cap N = \emptyset$. Suppose that there exists an $f \in (\sqrt{I} : s)[X] \cap N$. Then $sf \in \sqrt{I}[X]$ and $c(f) = R$; so $c(sf) \subseteq \sqrt{I}$, and thus $sR = c(sf) \subseteq \sqrt{I}$, a contradiction because $I \cap S = \emptyset$. Now, since $(\sqrt{I} : s)[X]$ is a prime ideal of $R[X]$ with $(\sqrt{I} : s)[X] \cap N = \emptyset$, $(\sqrt{I} : s)[X]_N$ is a prime ideal of $R[X]_N$. This implies that $(\sqrt{I[X]_N} :_{R[X]_N} s)$ is a prime ideal of $R[X]_N$, and hence $I[X]_N$ is a quasi-S-primary ideal of $R[X]_N$.

(2) ⇒ (1). Suppose that $I[X]_N$ is a quasi-S-primary ideal of $R[X]_N$. Then by [14, Remark 1], there exists an $s \in S$ such that $(\sqrt{I[X]_N} :_{R[X]_N} s)$ is a prime ideal of $R[X]_N$. This implies that $(\sqrt{I} : s)[X]_N$ is a prime ideal of $R[X]_N$. To complete the proof of this assertion it is sufficient to prove that $(\sqrt{I} : s)$ is a prime ideal of R. Let $a, b \in R$ such that $ab \in (\sqrt{I} : s)$. Since $(\sqrt{I} : s)[X]_N$ is a prime ideal of $R[X]_N$, $a \in (\sqrt{I} : s)[X]_N$ or $b \in (\sqrt{I} : s)[X]_N$. Assume that $a \in (\sqrt{I} : s)[X]_N$. Then $a = \frac{f}{g}$ for some $f \in (\sqrt{I} : s)[X]$ and some $g \in N$; so $ag = f \in (\sqrt{I} : s)[X]$ which implies that $aR = ac(g) = c(ag) \subseteq (\sqrt{I} : s)$. Thus $a \in (\sqrt{I} : s)$. In the same way we prove that if $b \in (\sqrt{I} : s)[X]_N$, then $b \in (\sqrt{I} : s)$. Hence $(\sqrt{I} : s)$ is a prime ideal of R.

(1) ⇔ (3). We proceed exactly as (1) ⇔ (2). ☐

Let $D \subseteq E$ be an extension of integral domains. Recall that a fractional ideal I of E is said to be *extended* from D if $I = JE$ for some ideal J of D. Also D is called a *Prüfer domain* if for every non-zero finitely generated ideal of D is invertible. It was shown in [2] that D is a Prüfer domain if and only if every ideal of $D[X]_N$ is extended from D, where $N = \{f \in D[X] \mid c(f) = D\}$. Using this result and

Theorem 7, we give a characterization of quasi-S-primary ideals of the Nagata ring $D[X]_N$, where D is a Prüfer domain.

Corollary 4. *Let D be a Prüfer domain and S a multiplicatively closed subset of D. Then the following statements are equivalent.*

(1) *Each proper ideal of D is quasi-S-primary.*

(2) *Each proper ideal of $D[X]_N$ is quasi-S-primary.*

We end this section by the following remark.

Remark 3. In [19, Corollary 2.27] it is shown that if every proper ideal of a ring R is S-prime, then R is a field. Is R a field if every proper ideal of a ring R is S-primary?

Let D be a one dimensional (Krull dimension) domain with a unique maximal ideal which is not a field (for example, a one-dimensional valuation domain) and S consists of units of D. Then every proper ideal of D is primary (S-primary). Indeed, the only prime ideals of D is (0) and M because $\dim(D) = 1$. On the other hand, let I be an ideal of D. We show that I is primary (S-primary). If $I = (0)$, then I is primary. If $I \neq (0)$, then M/I is the only prime ideal of the ring D/I. This implies that $(\bar{0})$ is a primary ideal of D/I (because $\sqrt{(\bar{0})} = M/I$). Hence I is a primary ideal of D. But D is not a field.

4. Quasi-S-primary ideals of amalgamation rings

In the classical ideal theory, pullbacks have for many years been an important tool in the commutative algebra because of their use in producing many examples. The $D + M$ construction by Gilmer and the Nagata's idealization are typical examples of pullbacks. (See [9, 12, 17]) In order to set up a more general setting of the idealization, D'Anna and Fontana introduced and studied the amalgamated duplication of a commutative ring R along an R-module M [5]. In [8], the authors developed the concept of the amalgamation equipped with an ideal by using a ring homomorphism. We start this section by giving a relationship between quasi-S-primary ideals of a ring R and those of the idealization ring $R(+)M$. First, let us recall the notion of idealization ring $R(+)M$. Let M be an R-module. We recall that $R(+)M = \{(r, m) : r \in R, m \in M\}$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity $(1, 0)$ and it is called the idealization of M. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. Moreover, the radical of $I(+)N$ is $\sqrt{I(+)N} = \sqrt{I}(+)M$. Note that if S is a multiplicatively closed subset of R, then $S(+)M$ and $S(+)0$ are multiplicatively closed subsets of $R(+)M$.

Proposition 10. *Let S be a multiplicatively closed subset of a ring R and M be an R-module. For an ideal I of R disjoint with S, the following statements are equivalent.*

(1) $I(+)M$ is a quasi-$S(+)M$-primary ideal of $R(+)M$.

(2) I is a quasi-S-primary ideal of R with a quasi-S-primary element $s \in S$.

(3) $I(+)M$ is a quasi-$S(+)0$-primary ideal of $R(+)M$.

Proof. (1)⇒(2). Suppose that (s, m) is a quasi-$S(+)M$-primary element of $I(+)M$ and $ab \in I$ for some $a, b \in R$. Then $(a, 0)(b, 0) \in I(+)M$, which implies either $(s, m)(a, 0) \in \sqrt{I(+)M}$ or $(s, m)(b, 0) \in \sqrt{I(+)M}$. Since $\sqrt{I(+)M} = \sqrt{I}(+)M$, we conclude either $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. Thus I is a quasi-S-primary ideal of R.

(2)⇒(3). Suppose that $(a, m_1)(b, m_2) \in I(+)M$ for some $(a, m_1), (b, m_2) \in R(+)M$. Then $ab \in I$ yields $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$, and thus $(s, 0)$ is a quasi-$S(+)0$-primary element of $I(+)M$.

(3)⇒(1). This follows from Remark 1 (4) as $S(+)0 \subseteq S(+)M$. □

Let R and R' be two rings, $f : R \to R'$ be a homomorphism, let I, J and K be ideals of the rings R, R' and $f(R)+J$, respectively. In [8], two types of ideals of $R \bowtie^f J$ are studied: $I \bowtie^f J = \{(i, f(i)+j) : i \in I, j \in J\}$ and $\overline{K}^f = \{(a, f(a)+j) : a \in R, j \in J, f(a)+j \in K\}$. Moreover, $\sqrt{I \bowtie^f J} = \sqrt{I} \bowtie^f J$ and $\sqrt{\overline{K}^f} = \overline{\sqrt{K}}^f$ [11, Lemma 8]. Whenever S is a multiplicatively closed subset of R, it can be easily shown that $S \bowtie^f J = \{(s, f(s)+j) : s \in S, j \in J\}$ and $W = \{(s, f(s)) : s \in S\}$ are multiplicatively closed subsets of $R \bowtie^f J$. For more detail regarding to amalgamated rings, the reader may refer to [6]-[8].

In the following, we investigate the equivalent statements for the ideal $I \bowtie^f J$ which is $(S \bowtie^f J)$–primary in $R \bowtie^f J$.

Theorem 8. *Consider the amalgamation of rings R and R' along the ideal J of R' with respect to a homomorphism $f : R \to R'$. Let S be a multiplicatively closed subset of R and I be an ideal of R disjoint with S. Then the following statements are equivalent.*

(1) $I \bowtie^f J$ is a quasi-W-primary ideal of $R \bowtie^f J$.

(2) $I \bowtie^f J$ is a quasi-$(S \bowtie^f J)$-primary ideal of $R \bowtie^f J$.

(3) I is a quasi-S-primary ideal of R.

Proof. Observe that $(S \bowtie^f J) \cap (I \bowtie^f J) = \emptyset$ if and only if $W \cap (I \bowtie^f J) = \emptyset$ if and only if $S \cap I = \emptyset$.

(1)⇒(2). Since we have the inclusion $W \subseteq (S \bowtie^f J)$, we are done by Remark 1(3).

(2)⇒(3). Let $I \bowtie^f J$ be an $(S \bowtie^f J)$-primary ideal of $R \bowtie^f J$ and let $a, b \in R$ such that $ab \in I$. Then $(a, f(a))(b, f(b)) \in I \bowtie^f J$, which implies that there exists an element $(s, f(s)+j) \in S \bowtie^f J$ such that either $(s, f(s)+j)(a, f(a)) \in \sqrt{I \bowtie^f J}$ or $(s, f(s)+j)(b, f(b)) \in \sqrt{I \bowtie^f J}$. From the equality $\sqrt{I \bowtie^f J} = \sqrt{I} \bowtie^f J$, we conclude either $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$, as needed.

(3)⇒(1). Suppose that $(a, f(a)+j_1)(b, f(b)+j_2) \in I \bowtie^f J$ for some $(a, f(a)+j_1), (b, f(b)+j_2) \in R \bowtie^f J$. Then $ab \in I$, which follows that there exists an element $s \in S$ satisfying either $sa \in \sqrt{I}$ or $sb \in \sqrt{I}$. Thus, $(s, f(s))(a, f(a)+j_1) \in \sqrt{I} \bowtie^f J = \sqrt{I \bowtie^f J}$ or $(s, f(s))(b, f(b)+j_2) \in \sqrt{I} \bowtie^f J = \sqrt{I \bowtie^f J}$ and $I \bowtie^f J$ is a quasi-W-primary ideal of $R \bowtie^f J$. □

Let T be a multiplicatively closed subset of the ring R'. Then one can easily verify that the set $\bar{T}^f = \{(s, f(s)+j) : s \in R, j \in J, f(s)+j \in T\}$ is a multiplicatively closed subset of $R \bowtie^f J$.

Theorem 9. *Consider the amalgamation of rings R and R' along the ideals J of R' with respect to an epimorphism $f : R \to R'$. For an ideal K of R' and a multiplicatively closed subset T of R' disjoint with K, $\bar{K}^f$ is a quasi-$\bar{T}^f$-primary ideal of $R \bowtie^f J$ if and only if K is a quasi-T-primary ideal of R'.*

Proof. First, note that clearly $T \cap K = \phi$ if and only if $\bar{T}^f \cap \bar{K}^f = \phi$.

Let $\bar{K}^f$ is a quasi-$\bar{T}^f$-primary ideal of $R \bowtie^f J$ and $a' := f(a), b' := f(b) \in R'$ for $a, b \in R$ such that $a'b' \in K$, Then $(a, f(a)), (b, f(b)) \in R \bowtie^f J$ with $(a, f(a))(b, f(b)) = (ab, f(ab)) \in \bar{K}^f$. Hence there exists an element $(s, f(s)+j)$ of $\bar{K}^f$ such that either $(s, f(s)+j)(a, f(a)) = (sa, (f(s)+j)f(a)) \in \sqrt{\bar{K}^f}$ or $(s, f(s)+j)(b, f(b)) = (sb, (f(s)+j)f(b)) \in \sqrt{\bar{K}^f}$. The equality $\sqrt{\bar{K}^f} = \overline{\sqrt{K}}^f$ yields that $f(s)+j \in T$ and $(f(s)+j)f(a) \in \sqrt{K}$ or $(f(s)+j)f(b) \in \sqrt{K}$. Thus K is a quasi-T-primary ideal of R'.

Conversely, suppose that K is a quasi-T-primary ideal of R'. Let $(a, f(a)+j_1)(b, f(b)+j_2) \in \bar{K}^f$ for some $(a, f(a)+j_1), (b, f(b)+j_2) \in R \bowtie^f J$. Then $(f(a)+j_1)(f(b)+j_2) \in K$, which follows that there exists a $f(s) \in T$ such that either $f(s)(f(a)+j_1) \in \sqrt{K}$ or $f(s)(f(b)+j_2) \in \sqrt{K}$. Hence, $(s, f(s))(a, f(a)+j_1)) = (sa, f(sa)+j_1 f(s)) \in \overline{\sqrt{K}}^f$ or $(s, f(s))(b, f(b)+j_2)) \in \overline{\sqrt{K}}^f$, and thus we are done by $\overline{\sqrt{K}}^f = \sqrt{\bar{K}^f}$. □

References

[1] Anderson, D. D., Dumitrescu, T. (2002). S-Noetherian rings. *Commun. Algebra* 30:4407–4416. DOI: 10.1081/AGB-120013328.

[2] Anderson, D. D. (1976). Multiplication ideals, multiplication rings and the ring $R(X)$. *Can. J. Math.* 28:760–768.

[3] Arnold, J. T. (1973). Prime ideals in power series rings. In *Conference on Commutative Algebra (University of Kansas, Lawrence, Kansas). Lecture Notes in Mathematics*, Vol. 311. Berlin: Springer, pp. 17–25.

[4] Atiyah, M., Macdonald I. (1969). *Introduction to Commutative Algebra*. Reading: Addison-Wesley.

[5] D'Anna, M. C., Finocchiaro, A., Fontana, M. (2009). Amalgamated algebras along an ideal. In: Fontana, M., Kabbaj, S.-E., Olberding, B., Swanson, I., eds. *Commutative Algebra and its Applications.* Berlin: Walter de Gruyter, pp. 241–252. DOI: 10.1515/9783110213188.155

[6] D'Anna, M. C., Fontana, M. (2007). An amalgamated duplication of a ring along an ideal: The basic properties. *J. Algebra Appl.* 6(3):443–459. DOI: 10.1142/S0219498807002326.

[7] D'Anna, M. C., Fontana, M. (2007). The amalgamated duplication of a ring along a multiplicative-canonical ideal. *Ark. Mat.* 45(2):241–252. DOI: 10.1007/s11512-006-0038-1

[8] D'Anna, M. C., Finocchiaro, C. A., Fontana, M. (2010). Properties of chains of prime ideals in an amalgamated algebra along an ideal. *J. Pure Appl. Algebra* 214:1633–1641. DOI: 10.1016/j.jpaa.2009.12.008.

[9] El Khalfi, A., Kim, H., Mahdou, N. (2022). Amalgamation extension in commutative ring theory: a survey, *Moroccan J. Algebra Geom. Appl.* 1(1):139–182.

[10] Fuchs, L. (1947). On quasi-primary ideals. *Acta Univ. Szeged. Sect. Sci. Math.* 11:174–183.

[11] Issoual, M., Mahdou, N., Yetkin Celikel, E., Ozkirisci, N. A., S-1-absorbing primary submodules (submitted).

[12] Gilmer, R. W. (1972). *Multiplicative Ideal Theory*, Vol. 12. New York: Marcel Dekker.

[13] Hamed, A., Hizem, S. (2016). Modules satisfying the S-Noetherian property and S-ACCR. *Commun. Algebra.* 44:1941–1951. DOI: 10.1080/00927872.2015.1027377

[14] Hamed, A., Malek, A. (2020). S-prime ideals of a commutative ring. *Beitr. Algebra Geom.* 61(3):533–542. DOI: 10.1007/s13366-019-00476-5

[15] Kang, B. G. (1989). Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. *J. Algebra* 123:151–170. DOI: 10.1016/0021-8693(89)90040-9

[16] Lu, C. P. (1988). Modules satisfying ACC on a certain type of colons. *Pac. J. Math.* 131(2):303–318.

[17] Nagata, M. (1962). *Local Rings*, Tracts in Pure and Applied Mathematics, No. 13. New York and London: Interscience Publishers, a division of Wiley.

[18] Samiei, M., Moghimi, H. F. (2016). Weakly irreducible ideals. *J. Algebra Relat. Top.* 4(2):9–17.

[19] Sevim, E. S., Arabacı, I. T., Tekir, U., Koc, S. (2019). On S-prime submodules. *Turkish J. Math.* 43:1036–1046. DOI: 10.3906/mat-1808-50

[20] Visweswaran, S. (2021). Some results on S-primary ideals of a commutative ring. *Beitr. Algebra Geom.* 63(2):247–266. DOI: 10.1007/s13366-021-00580-5