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Quasi J-submodules

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Abstract: Let R be a commutative ring with identity and M be a unitary R-module. The aim of this paper is to extend the notion of quasi J-ideals of commutative rings to quasi J-submodules of modules. We call a proper submodule N of M a quasi J-submodule if whenever $r \in R$ and $m \in M$ such that $rm \in N$ and $r \notin (J(R)M : M)$, then $m \in M\text{-}rad(N)$. We present various properties and characterizations of this concept (especially in finitely generated faithful multiplication modules). Furthermore, we provide new classes of modules generalizing presimplifiable modules and justify their relation with (quasi) J-submodules. Finally, for a submodule N of M and an ideal I of R, we characterize the quasi J-ideals of the idealization ring $R(+)M$.

Key words: Quasi J-submodule, J-submodule, quasi J-ideal, quasi J-presimplifiable module, J-presimplifiable module

1. Introduction

All rings considered in this paper are commutative with identity elements, and all modules are unital. Let R be a ring and N be a submodule of an R-module M. By $Z(R)$, $reg(R)$, $N(R)$, $J(R)$, $Z(M)$ and $M\text{-}rad(N)$, we denote the set of zero-divisors of R, the set of regular elements in R, the nil radical of R, the Jacobian radical of R, the set of all zero divisors on M; i.e. $\{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}$ and the intersection of all prime submodules of M containing N, respectively. For submodules N of M, the residual of N by M, $(N : M)$ denotes the ideal $\{r \in R : rM \subseteq N\}$. In particular, the ideal $(0 : M)$ is called the annihilator of M. Moreover, if I is an ideal of R, then the residual submodule N by I is $[N :_M I] = \{m \in M : Im \subseteq N\}$. An R-module M is a multiplication module if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = (N : M)M$, [7].

In 2015, Mohamadian [14] introduced the concept of r-ideals in commutative rings. A proper ideal I of a ring R is called an r-ideal if whenever $a, b \in R$, $ab \in I$ and $Ann(a) = 0$ imply that $b \in I$ where $Ann(a) = \{r \in R : ra = 0\}$. As a subclass of r-ideals, Tekir et al. [18] defined a proper ideal I of R to be n-ideal if whenever $a, b \in R$ such that $ab \in I$ and $a \notin N(R)$, then $b \in I$. Later, Khashan and Bani-Ata [10] generalized this notion to J-ideals and J-submodules. A proper ideal I of R is said to be a J-ideal if whenever $a, b \in R$ such that $ab \in I$ and $a \notin J(R)$, then $b \in I$. A proper submodule N of an R-module M is said to be a J-submodule if whenever $r \in R$ and $m \in M$ with $rm \in N$ and $r \notin (J(R)M : M)$, then $m \in N$. Generalizing

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the idea of J-ideals, as a very recent study [11], the class of quasi J-ideals has been defined and studied. A proper ideal I of R is said to be a quasi J-ideal if $\sqrt{I} = \{x \in R : x^n \in I$ for some $n \in \mathbb{Z}\}$ is a J-ideal.

The purpose of the present work is to generalize the notions of quasi J-ideals and J-submodules by defining and studying quasi J-submodules. We call a proper submodule N of M a quasi J-submodule if whenever $r \in R$ and $m \in M$ such that $rm \in N$ and $r \notin (J(R)M : M)$, then $m \in M\text{-}rad(N)$. In Section 2, we investigate many general properties of quasi J-submodules of an R-module M with various examples. J-submodules are obviously a quasi J-submodules, but the converse of this implication is not true in general (Example 2.2). Among many other results in this section, we investigate quasi J-submodules under various contexts of constructions such as homomorphic images, direct products and localizations (see Propositions 2.6, 2.8, and 2.10). In 1997, the notion of presimplifiable modules was first studied by Anderson and and Valdes-Leon [4] as R-modules with property that $Z(M) \subseteq J(R)$. Motivated from this concept, we introduce new generalizations of presimplifiable modules which are quasi presimplifiable, J-presimplifiable and quasi J-presimplifiable modules. Example 2.16 is given to show that these generalizations are proper. The main result of this section (Theorem 2.17) gives a relation between quasi J-submodules (resp. J-submodules) and quasi presimplifiable (resp. J-presimplifiable) modules which enables us to construct more examples for quasi J-submodules. Precisely, a submodule N of an R-module M contained in $J(R)M$ is a quasi J-submodule (resp. J-submodule) if and only if the quotient M/N is a nonzero quasi J-presimplifiable (resp. J-presimplifiable) R-module.

The last section deals with quasi J-submodules in multiplication modules. In Theorem 3.2, Theorem 3.4, and Proposition 3.5, we present many properties and characterizations for quasi J-submodules of multiplication modules (especially, in finitely generated faithful multiplication modules). In particular, in such a module M, we characterize quasi J-submodules N as those in which the residual ideal $(N : M)$ is a quasi J-ideal. It is shown in Theorem 3.14 that every quasi J-submodule of an R-module M is contained in a maximal quasi J-submodule of M. Furthermore, if M is finitely generated faithful multiplication, then a maximal quasi J-submodule of M is a J-submodule. For a submodule N of M and an ideal I of R, we finally (Theorem 3.18) give a characterization of J-ideals in the idealization ring $R(+)M$.

2. General properties of quasi J-submodules

In this section, among other results concerning the general properties of quasi J-submodules, some characterizations of this notion will be investigated. Moreover, the relations among quasi J-submodules and some other types of submodules will be clarified.

First, we present the fundamental definition of quasi J-submodules which will be studied in this paper.

Definition 2.1 *Let R be a ring and let M be an R-module. A proper submodule N of M is called a quasi J-submodule if whenever $r \in R$ and $m \in M$ such that $rm \in N$ and $r \notin (J(R)M : M)$, then $m \in M\text{-}rad(N)$.*

It is clear that any J-submodule of M is a quasi J-submodule. However, in the next example we can see that the converse is not true in general.

Example 2.2 *Consider the $\mathbb{Z}$-module $M = \mathbb{Z}_4$. Then one can directly see that $M\text{-}rad(\langle\bar{0}\rangle) = \langle\bar{2}\rangle$. Now, let $r \in \mathbb{Z}$ and $\bar{k} \in \mathbb{Z}_4$ such that $r \cdot \bar{k} \in \langle\bar{0}\rangle$ and $r \notin (J(\mathbb{Z})\mathbb{Z}_4 : \mathbb{Z}_4) = (0 : \mathbb{Z}_4) = \langle 4\rangle$. Then clearly*

$\bar{k} \in \langle \bar{2} \rangle = M - rad(\langle \bar{0} \rangle)$ and so $\langle \bar{0} \rangle$ is a quasi J-submodule. On the other hand, $\langle \bar{0} \rangle$ is not a J-submodule since for example, $2 \cdot \bar{2} = \bar{0}$ but $2 \notin (0 : \mathbb{Z}_4)$ and $\bar{2} \neq \bar{0}$.

Following [12], a proper submodule N of an R-module M is called an r-submodule (resp. sr-submodule) if whenever $am \in N$ with $ann_M(a) = 0_M$ (resp. $ann_R(m) = 0_R$), then $m \in N$ (resp. $a \in (N : M)$) for each $a \in R$ and $m \in M$.

In general, the class of (quasi) J-submodules is not comparable with the classes of r-submodules, sr-submodules and prime submodules.

Example 2.3 *(1) The submodule $N = \langle \bar{0} \rangle$ is a quasi J-submodule of the $\mathbb{Z}$-module $\mathbb{Z}_4$ which is not an r-submodule, an sr-submodule or a prime submodule.*

(2) The submodule $\langle 2 \rangle$ is a prime submodule of the $\mathbb{Z}$-module $\mathbb{Z}$ which is not a (quasi) J-submodule.

(3) The submodule $N = \langle \bar{2} \rangle$ is an r-submodule and sr-submodule of the $\mathbb{Z}$-module $\mathbb{Z}_6$, [12, Example 1] which is not a (quasi) J-submodule.

In the following result, we give a characterization for quasi J-submodules of an R-module M.

Proposition 2.4 *Let M be an R-module and N be a proper submodule of M. The following are equivalent:*

1. *N is a quasi J-submodule of M.*
2. *If $r \in R - (J(R)M : M)$ and K is a submodule of M with $rK \subseteq N$, then $K \subseteq M$-$rad(N)$.*
3. *If $A \nsubseteq (J(R)M : M)$ is an ideal of R and K is a submodule of M with $AK \subseteq N$, then $K \subseteq M$-$rad(N)$.*

Proof (1)$\Rightarrow$(2) Assume N is a quasi J-submodule. Suppose $r \in R - (J(R)M : M)$ and K is a submodule of M with $rK \subseteq N$. Then for all $k \in K$, $rk \in N$ and so $k \in M$-$rad(N)$. Therefore, $K \subseteq M$-$rad(N)$ as needed.

(2)$\Rightarrow$(3) Suppose $AK \subseteq N$ for an ideal $A \nsubseteq (J(R)M : M)$ and a submodule K of M. Then for $r \in A - (J(R)M : M)$, we have $rK \subseteq N$ and so by assumption, $K \subseteq M$-$rad(N)$.

(3)$\Rightarrow$(1) Let $r \in R$ and $m \in M$ such that $rm \in N$ and $r \notin (J(R)M : M)$. Then $\langle r \rangle \langle m \rangle \subseteq N$ and $\langle r \rangle \nsubseteq (J(R)M : M)$ and so $m \in \langle m \rangle \subseteq M$-$rad(N)$ and we are done. □

Lemma 2.5 *[13]Let $\varphi : M_1 \longrightarrow M_2$ be an R-module epimorphism.Then*

1. *If N is a submodule of M_1 and $\ker(\varphi) \subseteq N$, then $\varphi(M_1\text{-}rad(N)) = M_2\text{-}rad(\varphi(N))$.*
2. *If K is a submodule of M_2, then $\varphi^{-1}(M_2\text{-}rad(K)) = M_1\text{-}rad(\varphi^{-1}(K))$.*

Proposition 2.6 *Let $\varphi : M_1 \longrightarrow M_2$ be an R-module epimorphism. Then*

1. *If N is a quasi J-submodule of M_1 with $\ker(\varphi) \subseteq N$, then $\varphi(N)$ is a quasi J-submodule of M_2.*
2. *If K is a quasi J-submodule of M_2 with $\ker(\varphi) \subseteq J(R)M_1$, then $\varphi^{-1}(K)$ is a quasi J-submodule of M_1.*

Proof (1) Suppose $\varphi(N) = M_2 = \varphi(M_1)$ and let $m_1 \in M_1$. Then $\varphi(m_1) = \varphi(n)$ for some $n \in N$ and so $m_1 - n \in \ker(\varphi) \subseteq N$. So, $m_1 \in N$ and $N = M_1$ which is a contradiction. Hence, $\varphi(N)$ is proper in M_2. Let $r \in R$ and $m_2 \in M_2$ such that $rm_2 \in \varphi(N)$ and $r \notin (J(R)M_2 : M_2)$. Choose $m_1 \in M_1$ such that $\varphi(m_1) = m_2$. Then $rm_2 = r\varphi(m_1) = \varphi(rm_1) \in \varphi(N)$. Thus, $\varphi(rm_1 - a) = 0$ for some $a \in N$ and so $rm_1 - a \in \ker(\varphi) \subseteq N$. It follows that $rm_1 \in N$. Moreover, we have $r \notin (J(R)M_1 : M_1)$. Indeed, if $rM_1 \subseteq J(R)M_1$, then $rM_2 = r\varphi(M_1) = \varphi(rM_1) \subseteq \varphi(J(R)M_1) = J(R)\varphi(M_1) = J(R)M_2$ which is a contradiction. Since N is a quasi J-submodule, then $m_1 \in M_1\text{-}rad(N)$. Thus, $m_2 = \varphi(m_1) \in \varphi(M_1\text{-}rad(N)) = M_2\text{-}rad(\varphi(N))$ by Lemma 2.5 and $\varphi(N)$ is a quasi J-submodule of M_2.

(2) Clearly, $\varphi^{-1}(K)$ is proper in M_1. Let $r \in R$ and $m_1 \in M_1$ such that $rm_1 \in \varphi^{-1}(K)$ and $r \notin (J(R)M_1 : M_1)$. Then $r\varphi(m_1) = \varphi(rm_1) \in K$. We prove that $r \notin (J(R)M_2 : M_2)$. Suppose on the contrary that $rM_2 \subseteq J(R)M_2$. Then $r\varphi(M_1) \subseteq J(R)\varphi(M_1)$ and so $\varphi(rM_1) \subseteq \varphi(J(R)M_1)$. Now, if $x \in rM_1$, then $\varphi(x) \in \varphi(rM_1) \subseteq \varphi(J(R)M_1)$ and hence $x - t \in \ker(\varphi) \subseteq J(R)M_1$ for some $t \in J(R)M_1$. It follows that $x \in J(R)M_1$ and $rM_1 \subseteq J(R)M_1$ which is a contradiction. Since K is a quasi J-submodule of M_2, then $\varphi(m_1) \in M_2\text{-}rad(K)$. Therefore, $m_1 \in \varphi^{-1}(M_2\text{-}rad(K)) = M_1\text{-}rad(\varphi^{-1}(K))$ by Lemma 2.5 and the result follows. □

Corollary 2.7 *Let N and L be submodules of an R-module M with $L \subseteq N$. If N is a quasi J-submodule of M, then N/L is a quasi J-submodule of M/L.*

Proposition 2.8 *Let $M_1, M_2, ..., M_k$ be R-modules and consider the R-module $M = M_1 \times M_2 \times \cdots \times M_k$.*

(1) If $N = N_1 \times N_2 \times \cdots \times N_k$ is a quasi J-submodule of M, then N_i is a quasi J-submodule of M_i for all i such that $N_i \neq M_i$.

(2) If N_j is a quasi J-submodule of M_j for some $j \in \{1, 2, ..., k\}$, then $N = M_1 \times M_2 \times \cdots \times N_j \times \cdots \times M_k$ is a quasi J-submodule of M.

Proof (1) With no loss of generality, we assume $N_1 \neq M_1$ and prove that N_1 is a quasi J-submodule of M_1. Let $r \in R$ and $m \in M_1$ such that $rm \in N_1$ and $r \notin (J(R)M_1 : M_1)$. Then $r.(m, 0, ..., 0) \in N$ and clearly $r \notin (J(R)M : M)$. It follows that $(m, 0, ..., 0) \in M\text{-}rad(N)$ and so $m \in M\text{-}rad(N_1)$ as required.

(2) With no loss of generality, suppose N_1 is a quasi J-submodule of M_1. Let $r \in R$ and $(m_1, m_2, ..., m_k) \in M_1 \times M_2 \times \cdots \times M_k$ such that $(r.m_1, r.m_2, ..., r.m_k) = r.(m_1, m_2, ..., m_k) \in N_1 \times M_2 \times \cdots \times M_k$ and $r \notin (J(R)M : M)$. Then $rm_1 \in N_1$ and clearly $r \notin (J(R)M_1 : M_1)$. Therefore, $m_1 \in M_1\text{-}rad(N_1)$ and then $(m_1, m_2, ..., m_k) \in M\text{-}rad(N_1 \times M_2 \times \cdots \times M_k)$. □

Remark 2.9 *(1) If N_1 and N_2 are quasi J-submodules of R-modules M_1 and M_2 respectively, then $N_1 \times N_2$ need not be a quasi J-submodule of $M_1 \times M_2$. For example $\bar{0}$ and 0 are quasi J-submodules of the $\mathbb{Z}$-modules $\mathbb{Z}_4$ and $\mathbb{Z}$ respectively. However, $\bar{0} \times 0$ is not a quasi J-submodule of $\mathbb{Z}_4 \times \mathbb{Z}$ as $4.(\bar{1}, 0) \in \bar{0} \times 0$ but $4 \notin (\bar{0} \times 0 : \mathbb{Z}_4 \times \mathbb{Z})$ and $(\bar{1}, 0) \notin M\text{-}rad(\bar{0} \times 0) = 2\mathbb{Z}_4 \times 0$.*

(2) The condition $\ker(\varphi) \subseteq N$ in (1) of Proposition 2.6 is not necessary. Indeed, let M_1 and M_2 be R-modules and $\varphi : M_1 \times M_2 \to M_1$ be the projection epimorphism. If N_1 and N_2 are proper submodules of M_1 and M_2 and $N_1 \times N_2$ is a quasi J-submodule of $M_1 \times M_2$, then $\varphi(N_1 \times N_2) = N_1$ is a quasi J-submodule of M_1. However, $\ker(\varphi) = 0 \times M_2 \nsubseteq N_1 \times N_2$.

Let I be a proper ideal of R and N be a submodule of an R-module M. In the following proposition, the notations $Z_I(R)$ and $Z_N(M)$ denote the sets $\{r \in R : rs \in I$ for some $s \in R \backslash I\}$ and $\{r \in R : rm \in N$ for some $m \in M \backslash N\}$.

Proposition 2.10 *Let S be a multiplicatively closed subset of a ring R such that $S^{-1}(J(R)) = J(S^{-1}R)$ and M be an R-module. Then*

1. *If N is a quasi J-submodule of M and $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a quasi J-submodule of the $S^{-1}R$-module $S^{-1}M$.*
2. *If $S^{-1}N$ is a quasi J-submodule of $S^{-1}M$ and $S \cap Z_{(J(R)M:M)}(R) = S \cap Z_{M\text{-}rad(N)}(M) = \emptyset$, then N is a quasi J-submodule of M.*

Proof (1) Suppose that $\frac{r}{s_1}\frac{m}{s_2} \in S^{-1}N$. Then $urm \in N$ for some $u \in S$. Since N is a quasi J-submodule, then either $ur \in (J(R)M : M)$ or $m \in M\text{-}rad(N)$. If $ur \in (J(R)M : M)$, then $\frac{r}{s_1} = \frac{ur}{us_1} \in S^{-1}(J(R)M : M) = (S^{-1}J(R)\ S^{-1}M : S^{-1}M) = (J(S^{-1}R)\ S^{-1}M : S^{-1}M)$. If $m \in M\text{-}rad(N)$, then $\frac{m}{s_2} \in S^{-1}(M\text{-}rad(N)) = S^{-1}M\text{-}rad(S^{-1}N)$ and we are done.

(2) Let $r \in R$, $m \in M$ and $rm \in N$. Then $\frac{r}{1}\frac{m}{1} \in S^{-1}N$ which implies that $\frac{r}{1} \in (J(S^{-1}R)S^{-1}M : S^{-1}M) = S^{-1}(J(R)M : M)$ or $\frac{m}{1} \in S^{-1}M\text{-}rad(S^{-1}N) = S^{-1}(M\text{-}rad(N))$. Hence, either $ur \in (J(R)M : M)$ for some $u \in S$ or $vm \in M\text{-}rad(N)$ for some $v \in S$. Thus, our assumptions imply that either $r \in (J(R)M : M)$ or $m \in M\text{-}rad(N)$ as needed. □

Following [9], a proper submodule N of an R-module M is called quasi primary if whenever $r \in R$ and $m \in M$ such that $rm \in N$, then either $r \in \sqrt{N : M}$ or $m \in M\text{-}rad(N)$.

Proposition 2.11 *If N is a quasi-primary submodule of an R-module M such that $(N : M) \subseteq J(R)$, then N is a quasi J-submodule of M.*

Proof Suppose N is quasi-primary and $(N : M) \subseteq J(R)$. Let $r \in R$ and $m \in M$ such that $rm \in N$ and $r \notin (J(R)M : M)$. Then $r \notin J(R)$ and so by assumption $r \notin \sqrt{N : M}$. It follows that $m \in M\text{-}rad(N)$ as needed. □

Note that if $(N : M) \nsubseteq J(R)$, then the above proposition need not be true. For example, consider the submodule $N = \langle 2 \rangle$ of the $\mathbb{Z}$-module $\mathbb{Z}$. Then $(N : M) = \langle 2 \rangle \nsubseteq J(\mathbb{Z})$. Moreover, N is primary (and so quasi-primary) which is clearly not a quasi J-submodule. In view of [11, Theorem 2], we have:

Corollary 2.12 *If N is a quasi-primary submodule of an R-module M such that $(N : M)$ is a quasi J-ideal of R, then N is a quasi J-submodule of M.*

Following [16], a submodule N of an R-module M is called a pure submodule if $rM \cap N = rN$ for each $r \in R$. Moreover, N is called divisible if $rN = N$ for each $r \in Reg(R)$, the set of regular elements in R.

Proposition 2.13 *Let N be a divisible J-submodule of an R-module M with $(J(R)M : M) \subseteq Reg(R)$. Then N is pure in M.*

Proof It is clear that $rN \subseteq rM \cap N$ for each $r \in R$. Let $r \in R$ and let $n \in rM \cap N$. Then $n = rm \in N$ for some $m \in M$. If $r \in (J(R)M : M)$, then by assumption, $rM \cap N \subseteq N = rN$. If $r \notin (J(R)M : M)$, then $m \in N$ since N is a J-submodule of M and so $n = rm \in rN$. Thus, $rM \cap N = rN$ and N is pure in M. □

Synonymously to the Prime Avoidance Lemma for prime submodules, we have:

Proposition 2.14 *Let M be an R-module such that $J(R) = (J(R)M : M)$ is a quasi J-ideal of R. Let $N, N_1, N_2, ..., N_k$ be submodules of M where $N \subseteq \bigcup_{i=1}^{k} N_i$. Suppose that N_j is a J-submodule (resp. quasi J-submodule) with $(N_i : M) \nsubseteq J(R)$ for all $i \neq j$. If $N \nsubseteq \bigcup_{i \neq j}^{k} N_i$, then $N \subseteq N_j$ (resp. $N \subseteq M$-$rad(N_j)$).*

Proof Without loss of generality, assume that $j = k$. First, we show that $N \cap \left(\bigcap_{i=1}^{k-1} N_i\right) \subseteq N_k$. Let $x \in N \cap \left(\bigcap_{i=1}^{k-1} N_i\right)$. Since $N \nsubseteq \bigcup_{i=1}^{k-1} N_i$, there exists an element $m \in N_k$ but $m \notin \bigcup_{i=1}^{k-1} N_i$. Then clearly $m + x \in N \backslash \left(\bigcup_{i=1}^{k-1} N_i\right)$. Hence, $m + x \in N_k$ and so $x \in N_k$. Now, since $(N_i : M) \nsubseteq J(R)$ for all $i \neq k$, there is an element $r_i \in (N_i : M) \backslash J(R)$ for all $i \neq k$. Put $r = \prod_{i=1}^{k-1} r_i$. Since $J(R)$ is a prime ideal of R, [11, Corollary 2], then $r \notin J(R)$. Put $I = \bigcap_{i=1}^{k-1} (N_i : M)$. Then $I \nsubseteq J(R) = (J(R)M : M)$ and $IN \subseteq N \cap \left(\bigcap_{i=1}^{k-1} N_i\right) \subseteq N_k$. Since N_k is a (quasi) J-submodule, we conclude that $N \subseteq N_j$ (resp. $N \subseteq M$-$rad(N_j)$) by Proposition 2.4. □

For an R-module M, consider the set of all zero divisors on M, $Z(M) = \{r \in R : rm = 0 \text{ for some } 0 \neq m \in M\}$. Following [4], we call an R-module M presimplifiable if whenever $r \in R$ and $m \in M$ such that $rm = m$, then $m = 0$ or $r \in U(R)$. Equivalently, M is presimplifiable if and only if $Z(M) \subseteq J(R)$. We recall that the prime radical of an R-module M is the intersection of all prime submodules in M and is denoted by $Nil(M)$. It is known that for a submodule N of M, there is a one to one correspondence between the prime submodules of M/N and those of M containing N. Hence, we get $m \in M$-$rad(N) \iff m + N \in Nil(M/N)$.

More generally, let $NZ(M) = \{r \in R : rm = 0 \text{ for some } m \notin Nil(M)\}$. Next, we define some generalizations of presimplifiable modules.

Definition 2.15 *Let M be an R-module.*

1. *M is called quasi presimplifiable if $NZ(M) \subseteq J(R)$.*
2. *M is called J-presimplifiable if $Z(M) \subseteq (J(R)M : M)$.*
3. *M is called a quasi J-presimplifiable if $NZ(M) \subseteq (J(R)M : M)$.*

Example 2.16 *1. Consider the* $\mathbb{Z}$*-module* $M = \mathbb{Z}_p$ *for a prime integer* p*. Then* $Z(M) = \langle p \rangle = (0 : M) = (J(\mathbb{Z})M : M)$*. Thus,* M *is a* J*-presimplifiable module that is not presimplifiable.*

2. *The* $\mathbb{Z}$*-module* $M = \mathbb{Z}_4$ *is a quasi* J*-presimplifiable that is not* J*-presimplifiable. Indeed, we have* $Z(M) = 2\mathbb{Z}$ *and* $NZ(M) = 4\mathbb{Z} = (J(\mathbb{Z})M : M)$*.*

3. *Consider the* $\mathbb{Z}(+)\mathbb{Z}_2$*-module* $M = \mathbb{Z}(+)\mathbb{Z}_2$*. Then* M *is a quasi presimplifiable module that is not presimplifiable, see [11, Example 5].*

In the following theorem, we characterize J-submodules (resp. quasi J-submodules) in terms of J-presimplifiable (resp. quasi J-presimplifiable) modules.

Theorem 2.17 *Let* N *be a submodule of an* R*-module* M *with* $N \subseteq J(R)M$*. Then* N *is a quasi* J*-submodule (resp.* J*-submodule) if and only if* M/N *is a nonzero quasi* J*-presimplifiable (resp.* J*-presimplifiable)* R*-module.*

Proof Suppose N is a quasi J-submodule. Let $r \in NZ(M/N))$ and choose $m + N \notin Nil(M/N)$ such that $r(m + N) = N$. Then $rm \in N$ and $m \notin M\text{-}rad(N)$ since otherwise, if $m \in M\text{-}rad(N)$, then $m+N \in Nil(M/N)$, a contradiction. Since N is a quasi J-submodule, then $r \in (J(R)M : M) = ((J(R)M)/N : M/N) = (J(R)(M/N) : M/N)$ as needed. Conversely, suppose M/N is a nonzero quasi J-presimplifiable and let $r \in R$ and $m \in M$ such that $rm \in N$ and $r \notin (J(R)M : M) = (J(R)(M/N) : M/N)$. Then $r.(m+N) = N$ and $r \notin NZ(M/N)$. Therefore, we must have $m + N \in Nil(M/N)$ and then $m \in M\text{-}rad(N)$. The proof of the J-submodule case is similar. □

Corollary 2.18 *Let* N *be a submodule of an* R*-module* M *such that* $N \subseteq J(R)M$ *and* $(J(R)M : M) = J(R)$*. Then the following statements are equivalent:*

1. N *is a (quasi)* J*-submodule.*

2. M/N *is a nonzero (quasi)* J*-presimplifiable.*

3. M/N *is a nonzero (quasi) presimplifiable.*

Recall that for an R-module M, $T(M) = \{m \in M : ann_R(m) \neq 0\}$. It is clear that if N is an sr-submodule of M, then $N \subseteq T(M)$.

Proposition 2.19 *Let* M *be an* R*-module.*

(1) If M *is* J*-presimplifiable and* N *is an* r*-submodule of* M*, then* N *is a (quasi)* J*-submodule of* M*.*

(2) If N *is an* sr*-submodule of* M *with* $T(M) = N \subseteq J(R)M$*, then* N *is a (quasi)* J*-submodule of* M*.*

Proof (1) Suppose N is an r-submodule and let $r \in R$ and $m \in M$ such that $rm \in N$ and $r \notin (J(R)M : M)$. Since M is J-presimplifiable, then $r \notin Z(M)$ and so clearly $ann_M(r) = 0$. Hence, $m \in N \subseteq M\text{-}rad(N)$ as N is an r-submodule and we are done.

(2) Suppose N is an sr-submodule of M with $N = T(M)$. Let $r \in R$ and $m \in M$ such that $rm \in N$. If $m \notin M\text{-}rad(N)$, then $m \notin T(M)$ and so $ann_R(m) = 0$. By assumption, we get $r \in (N : M) \subseteq (J(R)M : M)$ and N is a (quasi) J-submodule of M. □

3. Quasi J-submodules in multiplication modules

In this section we study quasi J-submodules in some special types of modules. We give several properties and characterizations of quasi J-submodules in finitely generated faithful multiplication modules. Moreover, we determine conditions on a submodule N of M and an ideal I of R for which $I(+)N$ is a quasi J-ideal in $R(+)M$.

We start by the following lemma.

Lemma 3.1 *[17]Let M be a finitely generated faithful multiplication R-module, N be a proper submodule of M and I be an ideal of R. Then*

1. *M-$rad(N) = \sqrt{N : M}M$.*
2. *$IM : M = I$.*
3. *$(IN : M) = I(N : M)$.*

In view of the properties in Lemma 3.1, we give the following characterizations of quasi J-submodules of finitely generated faithful multiplication modules.

Theorem 3.2 *Let I be an ideal of a ring R and N be a submodule of a finitely generated faithful multiplication R-module M. Then*

1. I is a quasi J-ideal of R if and only if IM is a quasi J-submodule of M.
2. N is a quasi J-submodule of M if and only if $(N : M)$ is a quasi J-ideal of R.
3. N is a quasi J-submodule of M if and only if $N = IM$ for some quasi J-ideal of R.
4. If I is a quasi J-ideal of R and N is a quasi J-submodule of M, then IN is a quasi J-submodule of M.

Proof (1) Suppose I is a quasi J-ideal of R. If $IM = M$, then $I = (IM : M) = R$, a contradiction. Thus, IM is proper in M. Now, let $r \in R$ and $m \in M$ such that $rm \in IM$ and $r \notin (J(R)M : M) = J(R)$. Then $r((m) : M) = ((rm) : M) \subseteq (IM : M) \subseteq (\sqrt{I}M : M) = \sqrt{I}$. As I is a quasi J-ideal of R, we conclude that $((m) : M) \subseteq \sqrt{I}$. Thus, $m \in ((m) : M)M \subseteq \sqrt{I}M = M$-$rad(IM)$. Conversely, suppose IM is a quasi J-submodule of M. Then clearly I is proper in R. Let $a, b \in R$ such that $ab \in I$ and $a \notin J(R) = (J(R)M : M)$. Since $abM \in IM$ and IM is a quasi J-submodule, then $bM \subseteq M$-$rad(IM) = \sqrt{I}M$. Therefore, $b \in \sqrt{I}M : M = \sqrt{I}$ and I is a quasi J-ideal of R.

(2) Follows by (1) since $N = (N : M)M$.

(3) Follows by choosing $I = (N : M)$ and using (2).

(4) Suppose I is a quasi J-ideal of R and N is a quasi J-submodule of M. Now, $(N : M)$ is a quasi J-ideal of R by (2) and so $I(N : M)$ is also quasi J-ideal by [11, Proposition 4]. Moreover, $IN = I(N : M)M$ is proper in M since otherwise, $I(N : M) = R$, a contradiction. By using (1), we conclude that IN is a quasi J-submodule of M. □

However, the equivalence in (2) of Theorem 3.2 cannot be achieved if M is not finitely generated faithful multiplication. For example, consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$ and the submodule $N = 2\mathbb{Z} \times 0$ of M. Then clearly $(N:M)=0$ is a quasi J-ideal of $\mathbb{Z}$, but N is not a quasi J-submodule of M. In fact, $2.(1,0)\in N$ but neither $2\in(J(\mathbb{Z})M:M)=0$ nor $(1,0)\in M\text{-}rad(N)=N$.

Proposition 3.3 *Let N be a submodule of a faithful multiplication R-module M. Let I be a finitely generated faithful multiplication ideal of R. Then*

1. *If IN is a J-submodule of M, then either I is a J-ideal of R or N is a J-submodule of M.*
2. *If $\sqrt{I}$ is a finitely generated multiplication ideal of R and $\sqrt{I}N$ is a quasi J-submodule of M, then either I is a quasi J-ideal of R or N is a quasi J-submodule of M.*

Proof (1) If $N=M$, then $(IN:M)=I(N:M)=IR=I$ is a J-ideal of R by [10, Corollary 3.4]. Suppose $N\subsetneq M$. Since I is finitely generated faithful multiplication, we have $N=(IN:_M I)$, [1, Lemma 2.4]. Hence, one can easily verify that $(N:M)=((IN:_M I):M)=(I(N:M):I)$. Let $a,b\in R$ such that $ab\in(N:M)$ and $a\notin J(R)$. Then $Iab\subseteq I(N:M)=(IN:M)$ and so $Ib\subseteq I(N:M)$ as $(IN:M)$ is a J-ideal. It follows that $b\in(I(N:M):I)=(N:M)$ and so $(N:M)$ is a J-ideal of R. The result follows again by [10, Corollary 3.4].

(2) If $N=M$, then $\sqrt{I}=\sqrt{I}(N:M)=(\sqrt{I}N:M)$ is a quasi J-ideal of R by (2) of Theorem 3.2. It follows clearly that I is a quasi J-ideal. Suppose $N\subsetneq M$ and note again by [1, Lemma 2.4] that $M\text{-}rad(N)=(\sqrt{I}(M\text{-}rad(N)):_M\sqrt{I})$. Let $rm\in N$ and $r\notin J(R)$ for $r\in R$ and $m\in M$. Then $\sqrt{I}rm\subseteq\sqrt{I}N$ and so $\sqrt{I}m\subseteq M\text{-}rad(\sqrt{I}N)=\sqrt{I}(M\text{-}rad(N))$. It follows that $m\in(\sqrt{I}(M\text{-}rad(N)):_M\sqrt{I})=M\text{-}rad(N)$ and N is a quasi J-submodule of M. □

Theorem 3.4 *Let N be a proper submodule of a finitely generated faithful multiplication R-module M. The following are equivalent:*

1. N is a quasi J-submodule.
2. $M\text{-}rad(N)$ is a quasi J-submodule.
3. $M\text{-}rad(N)$ is a J-submodule.
4. $(M\text{-}rad(N):_M\langle r\rangle)=M\text{-}rad(N)$ for all $r\notin J(R)$.

Proof (1)⇒(2) Suppose N is a quasi J-submodule and let $r\in R$ and $m\in M$ such that $rm\in M\text{-}rad(N)$ and $r\notin(J(R)M:M)=J(R)$. Then $rm\in\sqrt{N:M}M$ and so $r((m):M)=((rm):M)\subseteq(\sqrt{N:M}M:M)=\sqrt{N:M}$. Since $(N:M)$ is a quasi J-ideal by Theorem 3.2, then $((m):M)\subseteq\sqrt{N:M}$. It follows that $m\in((m):M)M\subseteq\sqrt{N:M}M=M\text{-}rad(N)$.

(2)⇒(3) It is straightforward as $M\text{-}rad(M\text{-}rad(N))=M\text{-}rad(N)$.

(3)⇒(4) Let $m\in(M\text{-}rad(N):_M\langle r\rangle)$. Then $rm\in M\text{-}rad(N)$ with $r\notin(J(R)M:M)$ and so $m\in M\text{-}rad(N)$ by our assumption (2). The other inclusion is clear.

(4)⇒(1) Suppose that $rm\in N$ and $r\notin(J(R)M:M)$. Then $rm\in M\text{-}rad(N)$ and $r\notin J(R)$ which imply that $m\in(M\text{-}rad(N):_M<r>)=M\text{-}rad(N)$. Thus, N is a quasi J-submodule. □

In general the equivalences in Theorem 3.4 need not be true if M is not finitely generated faithful multiplication. For example, while $\langle\bar{0}\rangle$ is a quasi J-submodule in the $\mathbb{Z}$-module $M = \mathbb{Z}_4$, $M\text{-}rad(\langle\bar{0}\rangle) = \langle\bar{2}\rangle$ is not a J-submodule since for example $2 \cdot \bar{1} \in \langle\bar{2}\rangle$ while $2 \notin (\bar{0} : \mathbb{Z}_4)$ and $\bar{1} \notin \langle\bar{2}\rangle$.

In view of Theorem 3.2 and Theorem 3.4, we also have:

Proposition 3.5 *Let M be a finitely generated faithful multiplication R-module. For any submodule N of M, the following statements are equivalent.*

1. N is a quasi J-submodule of M.
2. $\sqrt{(N : M)}$ is a J-ideal of R.
3. $\sqrt{(N : M)}$ is a quasi J-ideal of R.
4. $(N : M)$ is a quasi J-ideal of R.

Proposition 3.6 *Let N, K and L be submodules of an R-module M and I be an ideal of R with $I \nsubseteq (J(R)M : M)$. Then*

1. If K and L are quasi J-submodules of M with $IK = IL$, then $M\text{-}rad(K) = M\text{-}rad(L)$.
2. If IN is a quasi J-submodule of a finitely generated faithful multiplication module M, then N is a quasi J-submodule of M.

Proof (1) Suppose that $IK = IL$. Then $IK \subseteq L$ and $I \nsubseteq (J(R)M : M)$ imply that $K \subseteq M\text{-}rad(L)$ by Proposition 2.4 and $M\text{-}rad(K) \subseteq M\text{-}rad(M\text{-}rad(L)) = M\text{-}rad(L)$. Similarly, we conclude that $M\text{-}rad(L) \subseteq M\text{-}rad(K)$, so the equality holds.

(2) Let IN be a quasi J-submodule of M. Since $IN \subseteq IN$ and $I \nsubseteq (J(R)M : M)$, we conclude that $N \subseteq M\text{-}rad(IN)$. Hence, $M\text{-}rad(N) = M\text{-}rad(IN)$ which is clearly a J-submodule of M by Theorem 3.4. It follows again by Theorem 3.4, that N is a quasi J-submodule of M. □

Lemma 3.7 *[1]Let I be a faithful multiplication ideal of a ring R and M be a faithful multiplication R-module. Then*

1. *For every submodule N of IM, we have $(IM)\text{-}rad(N) = I(M\text{-}rad(N :_M I))$.*
2. *If N is a submodule of M and I is finitely generated, then $N = (IN :_M I)$.*

Proposition 3.8 *Let I be a faithful multiplication ideal of a ring R and M be a faithful multiplication R-module. If N is a quasi J-submodule of IM, then $(N :_M I)$ is a quasi J-submodule of M. Moreover, the converse is true if R is quasi-local.*

Proof Suppose N is a quasi J-submodule of IM. Then $N \subsetneq IM$ and so clearly,$(N :_M I) \subsetneq M$. Let $r \in R$ and $m \in M$ such that $rm \in (N :_M I)$ and $r \notin (J(R)M : M)$. Then $rmI \subseteq N$ and by Lemma 3.7 $r \notin (J(R)IM : IM)$, hence $mI \subseteq (IM)\text{-}rad(N) = I(M\text{-}rad(N :_M I))$. It follows by Lemma 3.7 that $m \in (I(M\text{-}rad(N :_M I)) : I) = M\text{-}rad(N :_M I)$. Therefore, $(N :_M I)$ is a quasi J-submodule of M. Now, suppose R is quasi-local and $(N :_M I)$ is a quasi J-submodule of M. Then clearly, N is proper in M and $I = \langle a \rangle$ is principal, see [3]. Let $r \in R$ and $m \in IM$ such that $rm \in N$ and $r \notin (J(R)IM : IM)$. Since $I = \langle a \rangle$, then we may write $m = am_1$ for some $m_1 \in M$. Hence, $rm_1 \in (N :_M I)$ and clearly $r \notin (J(R)M : M)$. So, $m_1 \in M\text{-}rad((N :_M I))$ as $(N :_M I)$ is a quasi J-submodule of M. Again by Lemma 3.7, we have $m = am_1 \in I(M\text{-}rad(N :_M I)) = (IM)\text{-}rad(N)$ and the result follows. □

A submodule N of an R-module M is said to be small (or superfluous) in M, abbreviated $N \ll M$, in case for any submodule K of M, $N + K = M$ implies $K = M$.

Proposition 3.9 *Every quasi J-submodules of a finitely generated faithful multiplication R-module is small.*

Proof Let N be a quasi J-submodule of an R-module M and K be a submodule of M with $N + K = M$. Then clearly $(N : M) + (K : M) = (N + K : M) = R$ and $(N : M)$ is a quasi J-ideal of R by Theorem 3.2. Hence $(K : M) = R$ by [11, Proposition 4] and so $K = M$ as desired. □

Let M be an R-module and N be a submodule of M. We denote the intersection of all maximal submodules of M by $J(M)$. In particular, by $J(N)$, we denote the intersection of all maximal submodules of M containing N. It is well known that if M is finitely generated faithful multiplication, then $J(M) = J(R)M$, [8]. In particular, we have $J(N) = J(N : M)M$.

In the next two theorems, we obtain more characterizations for quasi J-submodules in finitely generated faithful multiplication modules.

Theorem 3.10 *Let N be a J-submodule of a finitely generated faithful multiplication R-module M. Then the following statements are equivalent:*

1. N is a quasi J-submodule of M.
2. $N \subseteq J(M)$ and if whenever $r \in R$ and $m \in M$ with $rm \in N$ and $r \notin (J(N) : M)$, then $m \in M\text{-}rad(N)$.

Proof (1)⇒(2) Suppose N is a quasi J-submodule. since $(N : M)$ is a quasi J-ideal by Theorem 3.2, then $(N : M) \subseteq J(R)$, [11, Theorem 2]. Thus, $N = (N : M)M \subseteq J(R)M = J(M)$. Moreover, let $r \in R$ and $m \in M$ with $rm \in N$ and $r \notin (J(N) : M)$. Then $r \notin (J(M) : M) = (J(R)M : M)$ as clearly $J(M) \subseteq J(N)$ and so $m \in M\text{-}rad(N)$ by assumption.

(2)⇒(1) If $N = M$, then $J(M) = M$, a contradiction. Let $r \in R$ and $m \in M$ with $rm \in N$ and $r \notin (J(R)M : M) = J(R)$. Since $N \subseteq J(M)$, then one can easily see that $J(N) \subseteq J(J(M)) = J(M)$ and so $J(M) = J(N)$. Thus, $r \notin (J(N) : M)$ and so $m \in M\text{-}rad(N)$ as required. □

Recall that If M is a multiplication R-module and $N = IM$, $K = JM$ are two submodules of M, then the product NK of N and K is defined as $NK = (IM)(JM) = (IJ)M$. In particular, if $m_1, m_2 \in M$, then $m_1 m_2 = \langle m_1 \rangle \langle m_2 \rangle$.

Proposition 3.11 *Let M be a finitely generated faithful multiplication R-module and N a proper submodule of M. Then N is a quasi J-submodule of M if and only if whenever K and L are submodules of M with $KL \subseteq N$, then $K \subseteq J(M)$ or $L \subseteq M\text{-}rad(N)$.*

Proof Suppose $K = IM$ and $L = JM$ for some ideals I and J of R and $KL \subseteq N$. Then $I(JM) \subseteq N$ and so $I \subseteq (J(R)M : M) = J(R)$ or $L = JM \subseteq M\text{-}rad(N)$ by Proposition 2.4. Thus, $K \subseteq J(R)M = J(M)$ or

$L \subseteq M$-$rad(N)$. Conversely, let $A \nsubseteq (J(R)M : M) = J(R)$ be an ideal of R and L be a submodule of M with $AL \subseteq N$. Then the result follows by putting $K = AM$ and using again Proposition 2.4. □

Corollary 3.12 *Let N be a proper submodule of a finitely generated faithful multiplication R-module M. Then N is a quasi J-submodule of M if and only if whenever $m_1, m_2 \in M$ such that $m_1 m_2 \in N$, then $m_1 \in J(M)$ or $m_2 \in M$-$rad(N)$.*

Proposition 3.13 *Let M be a finitely generated faithful multiplication R-module and let S be a subset of R with $S \nsubseteq J(R)$. If N is a quasi J-submodule of M, then $(N :_M S)$ is a quasi J-submodule of M.*

Proof First, we prove that $(N :_M S)$ is proper in M. Suppose $(N :_M S) = M$ and let $m \in M$. Then $Sm \subseteq N$ and since N is a quasi J-submodule, we get $m \in M$-$rad(N)$. Thus, $M = M$-$rad(N) = N$, a contradiction. Now, similar to the proof of (4) in Theorem 3.4, one can prove that $(M - rad(N) :_M S) = M - rad(N)$. Suppose that $rm \in (N :_M S)$ and $r \notin J(R)$. Then $rSm \subseteq N$ and so $Sm \subseteq M - rad(N)$ as N is a quasi J-submodule. It follows that $m \in (M - rad(N) :_M S) = M - rad(N) \subseteq M - rad(N :_M S)$ as required. □

A proper submodule N of an R-module M is called a maximal quasi J-submodule if there is no quasi J-submodule which contains N properly.

Theorem 3.14 *Every quasi J-submodule of an R-module M is contained in a maximal quasi J-submodule of M. Moreover, if M is finitely generated faithful multiplication, then a maximal quasi J-submodule of M is a J-submodule.*

Proof Suppose that N is a quasi J-submodule of M and Set $\Omega = \{N_\alpha : N_\alpha$ is a quasi J-submodule of M, $\alpha \in \Lambda\}$. Then $\Omega \neq \emptyset$. Let $N_1 \subseteq N_2 \subseteq \cdots$ be any chain in Ω. We show that $\bigcup_{i=1}^{\infty} N_i$ is a quasi J-submodule of M. Suppose $rm \in \bigcup_{i=1}^{\infty} N_i$ for $r \in R$, $m \in M$ and $r \notin (J(R)M : M)$. Then $rm \in N_j$ for some $j \in \mathbb{N}$ which implies that $m \in M$-$rad(N_j) \subseteq M - rad(\bigcup_{i=1}^{\infty} N_i)$. Since also $\bigcup_{i=1}^{\infty} N_i$ is clearly proper, then $\bigcup_{i=1}^{\infty} N_i$ is a quasi J-submodule which is an upper bound of the chain $\{N_i : i \in \mathbb{N}\}$. By Zorn's Lemma, Ω has a maximal element which is a maximal quasi J-submodule of M. Now, let K be a maximal quasi J-submodule of M. Suppose that $rm \in K$ and $r \notin J(R)$. Then $(K :_M r)$ is also a quasi J-submodule of M by Proposition 3.13. Thus, the maximality of K implies that $m \in (K :_M r) = K$ and we are done. □

In view of Theorem 3.14, we have the following.

Corollary 3.15 *Let M be a finitely generated faithful multiplication R-module. Then the following statements are equivalent:*

1. *$J(M)$ is a J-submodule of M.*
2. *$J(M)$ is a quasi J-submodule of M.*
3. *$J(M)$ is a prime submodule of M.*

Proof (1)$\Rightarrow$ (2) is clear.

(2)$\Rightarrow$ (1) It follows since $J(M)$ is the unique maximal quasi J-submodule of M by Theorem 3.10.

(2)$\Leftrightarrow$ (3) Since $M - rad(J(M)) = J(M)$, the claim is clear. □

In the following proposition, we prove that J-submodule property passes to a finite intersection and product.

Proposition 3.16 *Let M be a multiplication R-module and $N_1, N_2, ..., N_k$ be quasi J-submodules of M. Then so are $\bigcap_{i=1}^{k} N_i$ and $\prod_{i=1}^{k} N_i$.*

Proof Suppose that $rm \in \bigcap_{i=1}^{k} N_i$ and $r \notin (J(R)M : M)$. Then $rm \in N_i$ for all $i = 1, ..., k$ which gives $m \in M\text{-}rad(N_i)$ for all $i = 1, ..., k$. Since $\bigcap_{i=1}^{k} M\text{-}rad(Ni) = M\text{-}rad\left(\bigcap_{i=1}^{k} N_i\right)$ [2, Theorem 15 (3)], we conclude that $\bigcap_{i=1}^{k} N_i$ is a quasi J-submodule. By using the similar argument and the equality $\prod_{i=1}^{k} M\text{-}rad(Ni) = M\text{-}rad\left(\bigcap_{i=1}^{k} N_i\right)$ [15, Proposition 2.14.], $\prod_{i=1}^{k} N_i$ is also a quasi J-submodules of M. □

The converse of the above proposition can be achieved under certain conditions.

Proposition 3.17 *Let M be a finitely generated faithful multiplication R-module and $N_1, N_2, ..., N_k$ be quasi primary submodules of M such that $\sqrt{N_i : M}$ are not comparable for all $i = 1, ..., k$. If $\bigcap_{i=1}^{k} N_i$ or $\prod_{i=1}^{k} N_i$ is a quasi J-submodule of M, then N_i is a quasi J-submodule of M for each $i = 1, ..., k$.*

Proof Suppose that N_i $(i = 1, ..., k)$ is a quasi primary submodule of M. Then $(N_i : M)$ is a quasi primary ideal of R for all $i = 1, ..., k$, [9, Lemma 2.12]. If $\bigcap_{i=1}^{k} N_i$ is a quasi J-submodule, we conclude from Theorem 3.2 that $\left(\bigcap_{i=1}^{k} N_i : M\right) = \bigcap_{i=1}^{k} (N_i : M)$ is a quasi J-ideal of R. Hence, $(N_i : M)$ is a quasi J-ideal of R for all $i = 1, ..., k$ by [11, Proposition 5]. Thus, N_i is a quasi J-submodule of M for all $i = 1, ..., k$ by Theorem 3.2. The proof of the finite product case is similar since $\left(\prod_{i=1}^{k} N_i : M\right) = \prod_{i=1}^{k} (N_i : M)$ and by using Theorem 3.2 and [11, Proposition 6]. □

Let R be a ring and M be an R-module. The idealization ring of M is the set $R(+)M = R \oplus M = \{(r, m) : r \in R,\ m_2 \in M\}$ with coordinate-wise addition and multiplication defined as $(r_1, m_1)(r_2 m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. If I is an ideal of R and N a submodule of M, then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. It is well known that if $I(+)N$ is an ideal of $R(+)M$, then $\sqrt{I(+)N} = \sqrt{I}(+)M$. Moreover, we have $J(R(+)M) = J(R)(+)M$, [6]. Next, we characterize quasi J-ideals in any idealization ring $R(+)M$.

Theorem 3.18 *Let I be an ideal of of a ring R and N be a submodule of an R-module M. Then $I(+)N$ is a quasi J-ideal of $R(+)M$ if and only if I is a quasi J-ideal of R.*

Proof Suppose $I(+)N$ is a quasi J-ideal of $R(+)M$ and let $a, b \in R$ such that $ab \in I$ and $a \notin J(R)$. Then $(a,0)(b,0) \in I(+)N$ and $(a,0) \notin J(R(+)M)$. Therefore, $(b,0) \in \sqrt{I(+)N} = \sqrt{I}(+)M$ and so $b \in \sqrt{I}$ as needed. Conversely, suppose I is a quasi J-ideal of R. Let $(r_1, m_1), (r_2, m_2) \in R(+)M$ such that $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1) \in I(+)N$ and $(r_1, m_1) \notin J(R(+)M) = J(R)(+)M$. Then $r_1 r_2 \in I$ and $r_1 \notin J(R)$ which imply that $r_2 \in \sqrt{I}$. Thus, $(r_2, m_2) \in \sqrt{I}(+)M = \sqrt{I(+)N}$ and $I(+)N$ is a quasi J-ideal of $R(+)M$. □

In view of Theorem 3.18, we have the following result.

Corollary 3.19 *Let I be an ideal of of a ring R and M be a finitely generated faithful multiplication R-module. If IM is a quasi J-submodule of M, then $I(+)N$ is a quasi J-ideal of $R(+)M$ for any submodule N of M.*

Proof The result follows by Theorem 3.18 and (1) of Theorem 3.2. □

We note that if $I(+)N$ is a quasi J-ideal of $R(+)M$, then N need not be a quasi J-submodule of M. For example, while $0(+)\bar{0}$ is a quasi J-ideal of $\mathbb{Z}(+)\mathbb{Z}_6$ by Theorem 3.18, but $\bar{0}$ is not quasi J-submodule of $\mathbb{Z}_6$. For example, $2.\bar{3} = \bar{0}$ but $2 \notin (J(\mathbb{Z})\mathbb{Z}_6 : \mathbb{Z}_6) = \langle 6 \rangle$ and $\bar{3} \notin M\text{-}rad(\bar{0}) = \bar{0}$.

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