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S-radical of an ideal and strongly S-n ideals

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Abstract

In this paper, we investigate some properties of S-radical which is a generalization of classical radical. We moreover, determine S-radical of several types of ideals. Then we introduce the notion of strongly S-n-ideals that is an intermediate class between n-ideals and S-n-ideals. We investigate many properties of this class of ideals with illustrative examples. Finally, we obtain some results related to strongly S-n-ideals in amalgamated algebra.

Keywords n-ideal · S-n-ideal · S-prime ideal · S-radical

Mathematics Subject Classification: 13C13 · 13H99 · 13E10

1 Introduction

In this article, we focus only on commutative rings with nonzero identity and nonzero unital modules. Let R always denote such a ring and M denote such an R-module. The set of all nilpotent elements, zero divisors and regular elements of R are denoted by $n(R)$, $zd(R)$ and $reg(R)$, respectively. Also, we denote the Jacobson radical of R which is the intersection of all maximal ideals of R by $Jac(R)$. In 2015, R. Mohamadian defined the concept of r-ideals in commutative rings as follows: a proper ideal P of R is said to be an r-ideal if whenever $ab \in P$ and $a \in reg(R)$ for some $a, b \in R$, then $b \in P$ [17]. In [17], the author characterized r-ideals in some important classes of rings such as integral domains, quasi-regular rings (rings whose total quotient rings

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are von Neumann regular) and rings satisfying Property R(that is: all finitely generated ideals consisting of zero divisors have nonzero annihilators). Afterwards, Tekir et al. studied a subclass of r-ideals which is called n-ideals [19]. A proper ideal P of R is said to be an *n-ideal* if $ab \in P$ and $a \notin n(R)$ for some $a, b \in R$ imply that $b \in P$. The authors showed that every n-ideal is also an r-ideal [19, Proposition 2.5]. But the converse is not true in general. For instance, let $R = \mathbb{Z}_6$. While $P = (\bar{3})$ is an r-ideal in R, it is not an n-ideal. Also, they used n-ideals to characterize certain rings such as integral domains, fields and rings whose Zariski topology is irreducible (namely, rings having unique minimal prime ideal). A ring R (not necessarily commutative) is said to be a *UN-ring* if every nonunit element in R can be written a product of a unit and a nilpotent element [6]. The authors, also in [19, Proposition 2.25], showed that a ring R is a UN-ring if and only if each of its proper ideals is an n-ideal. In [12], the authors introduced S-n-ideals which are generalizations of n-ideals. Let S be a multiplicatively closed subset of ring R. According to the paper [12], an ideal I disjoint with S is said to be an S-n-ideal of R if there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$ and $sa \notin \sqrt{0}$, then $sb \in I$. This fixed element $s \in S$ is called an S-element of I. That generalization is proper as we can built many examples for S-n-ideals of rings which are not n-ideals similar to [12, Example 1 and 2]. On the other hand, as a recent research, Yıldız et al. generalized the concept of the radical of an ideal in a ring to S-radical of an ideal in [22]. For an ideal I of a ring R, the S-radical of I is defined by $\sqrt[S]{I} = \{a \in R : sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\}$.

The main aim of this article is to introduce a new generalization of n-ideals which is also a subclass of S-n-ideals in terms of the S-radical. To achieve this goal, in Sect 2, we investigate many properties of the S-radical of an ideal. For example, in Theorem 2.2, we prove the equality $\sqrt[S]{I} = \bigcap\limits_{\substack{P\in Spec_S(R)\\ I\subseteq P}} (P : s_P)$ where $Spec_S(R)$ is the set of all S-prime ideals of R that we will define later. We totally determine S-radicals of the ideals in the rings $\mathbb{Z}$ and $\mathbb{Z}_n$ for any multiplicatively closed subset S. (see Theorems 2.7, 2.10). In Sect 3, by using the idea of S-radical, we introduce a new class of ideals, namely strongly S-n-ideal which lies between n-ideals and S-n-ideals. Let S be a multiplicatively closed subset of a ring R. We call a proper ideal I of R disjoint with S a strongly S-n-ideal if for $a, b \in R$, $ab \in I$ and $a \notin I$ imply $b \in \sqrt[S]{0}$. Among many results in this section, we give some characterizations of this class of ideals (see Proposition 3.4, Theorems 3.5 and 3.6)). We introduce a new characterization for UN-rings in terms of those ideals (see Proposition 3.8). Also, we determine all strongly S_p-n-ideals of the ring $\mathbb{Z}_m$ (see Theorem 3.9). As usual, we discuss the behavior of strongly S-n-ideals under homomorphism, localization, cartesian products of rings and the idealization rings (see Propositions 3.11, 3.16, 3.18). Furthermore, in the last section, we study the S-radicals and strongly S-n-ideals in amalgamation rings.

In the following table, we provide a summary of the definitions of several families of ideals in commutative rings that are essential to the discussions and results presented in this paper. These definitions will serve as the foundational concepts throughout the study, and understanding them is crucial for the comprehension of the subsequent analyses and proofs. Let I be a proper ideal of a ring R.

r-ideal	If whenever $ab \in I$ and $a \in reg(R)$ for some $a, b \in R$, then $b \in I$
n-ideal	If $ab \in I$ and $a \notin n(R)$ for some $a, b \in R$ imply that $b \in I$
S-n-ideal	If there exists an (fixed) $s \in S$ such that for all $a, b \in R$ if $ab \in I$ and $sa \notin \sqrt{0}$, then $sb \in I$
Strongly S-n-ideal	If for $a, b \in R$, $ab \in I$ and $a \notin I$ imply $b \in \sqrt[S]{0}$
UN-ring	If every nonunit element in R can be written a product of a unit and a nilpotent element
quasi-regular ring	Its total quotient rings are von Neumann regular
Property R	All finitely generated ideals consisting of zero divisors have nonzero annihilators

2 Some properties of S-radical

Let S be a multiplicatively closed subset (briefly, m.c.s.) of a ring R. The S-radical of an ideal I of R has been defined by Yıldız et al. in [22] during the construction of S-Zariski topology on the set $Spec_S(R)$ of all S-prime ideals of R. In this section, we study several properties of this concept.

Definition 2.1 [22] Let I be an ideal of a ring R and S be an m.c.s. of R. The S-radical of I is defined by

$$\sqrt[S]{I} = \{a \in R : sa^n \in I \text{ for some } s \in S \text{ and } n \in \mathbb{N}\},$$

Moreover, an ideal I of R is called an S-radical ideal if $\sqrt[S]{I} = I$.

Clearly, $\sqrt[S]{I}$ is an ideal of R containing $\sqrt{I}$. However, this containment is proper in general. For example, consider the m.c.s. $S = \{1, 2, 4, 8\}$ of $\mathbb{Z}_{12}$. Then $\sqrt[S]{\langle 6\rangle} = \langle 3\rangle \supsetneq \langle 6\rangle = \sqrt{\langle 6\rangle}$. Also, this example shows that while clearly, every S-radical ideal is radical, the converse is not true in general. One can easily observe that $\sqrt[S]{I}$ is proper in R if and only if $I \cap S = \emptyset$. Furthermore, it is clear that $a \in \sqrt[S]{I}$ if and only if $sa \in \sqrt{I}$ for some $s \in S$. The S-radical of the zero ideal $\sqrt[S]{0}$ is called the S-nilradical of R and denoted by $N_S(R)$. It is clear that if $S \subseteq T$ are multiplicatively closed subsets of R, then $\sqrt[S]{I} \subseteq \sqrt[T]{I}$.

Given an m.c.s. S of R, consider the natural homomorphism $\pi : R \to S^{-1}R$ defined by $\pi(a) = \frac{a}{1}$ for each $a \in R$. For any ideal I of R and any ideal J of $S^{-1}R$, the extension of I is denoted by $S^{-1}I = \{\lambda \in S^{-1}R : \lambda = \frac{x}{s} \text{ for some } x \in I, s \in S\}$ and the inverse image of J is denoted by J^c. We can see that $\sqrt[S]{I} = \sqrt{(S^{-1}I)^c}$. Indeed, $a \in \sqrt[S]{I} \Longleftrightarrow sa \in \sqrt{I}$ for some $s \in S \Longleftrightarrow \frac{a}{1} = \frac{as}{s} \in S^{-1}\sqrt{I} = \sqrt{S^{-1}I} \Longleftrightarrow a \in \left(\sqrt{S^{-1}I}\right)^c = \sqrt{(S^{-1}I)^c}$.

Let S be a multiplicatively closed subset of a ring R. Recall that an ideal P of a ring R with $P \cap S = \emptyset$ is called S-prime if there is an element $s_P \in S$ such that whenever $a, b \in R$, $ab \in I$ implies $s_P a \in P$ or $s_P b \in P$. In this case, s_P is called an S-prime element of P and $(P : s_P)$ is a prime ideal of R [11]. Also, recall from [22] that $(P : s) \subseteq (P : s_P)$ for each $s \in S$. The set of all S-prime ideals of R is denoted by $Spec_S(R)$. Moreover, the S-variety of an ideal I of R is defined as $V_S(I) = \{P \in Spec_S(R) : sI \subseteq P$ for some $s \in S\}$. The S-radical of I is characterized in [22, Proposition 5] as $\sqrt[S]{I} = \bigcap\limits_{P \in V_S(I)} (P : s_P)$ and in particular, $N_S(R) = \bigcap\limits_{P \in Spec_S(R)} (P : s_P)$. However, in the following proposition we see that the intersection can be taken over the subset $\{P \in Spec_S(R) : I \subseteq P\}$ of $V_S(I)$.

Theorem 2.2 *Let I be an ideal of a ring R and S be a multiplicatively closed subset of R such that $I \cap S = \emptyset$. Then* $\sqrt[S]{I} = \bigcap\limits_{\substack{P \in Spec_S(R)\\ I \subseteq P}} (P : s_P)$.

Proof Clearly, $\bigcap\limits_{P \in V_S(I)} (P : s_P) \subseteq \bigcap\limits_{\substack{P \in Spec_S(R)\\ I \subseteq P}} (P : s_P)$. Let $x \in \bigcap\limits_{\substack{P \in Spec_S(R)\\ I \subseteq P}} (P : s_P)$ and let $P \in Spec_S(R)$ such that $sI \subseteq P$ for some $s \in S$. Then $I \subseteq (P : s) \subseteq (P : s_P)$ where $(P : s_P)$ is a prime ideal of R that is a maximal element of $\left\{(P : s')\right\}_{s' \in S}$, [22, Proposition 2]. Now, clearly $(P : s_P) \in Spec_S(R)$ with $I \subseteq (P : s_P)$ and so $x \in ((P : s_P) : s_{(P:s_P)}) = (P : s_P)$. Therefore, $x \in \bigcap\limits_{P \in V_S(I)} (P : s_P)$ and

$$\sqrt[S]{I} = \bigcap_{P \in V_S(I)} (P : s_P) = \bigcap_{\substack{P \in Spec_S(R)\\ I \subseteq P}} (P : s_P). \qquad \square$$

We note that unless $S \subseteq U(R)$, $V_S(I)$ and $\{P \in Spec_S(R) : I \subseteq P\}$ are different in general.

Example 2.3 Consider the ideal $I = m\mathbb{Z} \times \{0\}$ of the ring $R = \mathbb{Z} \times \mathbb{Z}$ where $m \neq 0$ and let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. Since $s = (0, 1) \in S$ with $sI = (0, 0)$, then $V_S(I) = Spec_S(R)$. On the other hand, clearly, $\{0\} \times \mathbb{Z} \in V_S(I) \backslash \{P \in Spec_S(R) : I \subseteq P\}$.

In particular, the following corollary follows from Theorem 2.2.

Corollary 2.4 *If P is an S-prime ideal of a ring R, then $\sqrt[S]{P} = (P : s_P)$ for any S-prime element s_P of P.*

For an m.c.s. S of a ring R and an ideal I of R, we note that $S/I = \{s + I : s \in S\}$ is an m.c.s. of R/I. By $Z_R(S)$, we denote the ideal $\{r \in R : rs = 0$ for some $s \in S\}$ of R. In the following, we list some basic properties of the S-radical of ideals which are analogous to those of radicals of ideals.

Proposition 2.5 *Let R be a ring, S be an m.c.s. and I, J be ideals of R. Then*

1. $\sqrt[S]{I}$ is a radical ideal of R.
2. $\sqrt[S]{I}/I = N_{S/I}(R/I)$ and $\sqrt[S]{I}/\sqrt{I} = Z_{R/\sqrt{I}}(S/\sqrt{I})$.
3. $\sqrt[S]{I + J} = \sqrt[S]{\sqrt[S]{I} + \sqrt[S]{J}}$
4. $\sqrt[S]{\sqrt[S]{I}} = \sqrt[S]{I}$
5. $S^{-1}(\sqrt[S]{I} + \sqrt[S]{J}) = S^{-1}R$ iff $S^{-1}(\sqrt{I} + \sqrt{J}) = S^{-1}R$ iff $S^{-1}(I + J) = S^{-1}R$.

6. If $\sqrt[S]{I}+\sqrt[S]{J}=R$, then $S^{-1}(I+J)=S^{-1}R$.
7. $\sqrt[S]{IJ}=\sqrt[S]{I\cap J}=\sqrt[S]{I}\cap\sqrt[S]{J}$.

Proof (1) Follows from Theorem 2.2 as $\sqrt[S]{I}$ is an intersection of prime ideals of R.

(2) Let $x+I\in\sqrt[S]{I}/I$ and choose $s\in S$ such that $sx\in\sqrt{I}$. Then $(s+I)(x+I)=sx+I\in\sqrt{I}/I=N(R/I)$ and so $x+I\in\sqrt[S/I]{0_{R/I}}=N_{S/I}(R/I)$. The reverse inclusion can be achieved similarly. Next, let $x+\sqrt{I}\in\sqrt[S]{I}/\sqrt{I}$. Then $x\in\sqrt[S]{I}$ and $sx\in\sqrt{I}$ for some $s\in S$. Therefore, $(s+\sqrt{I})(x+\sqrt{I})=sx+\sqrt{I}=\sqrt{I}$ and so $x+\sqrt{I}\in Z_{R/\sqrt{I}}(S/\sqrt{I})$. Also, the reverse inclusion is similar.

(3) Since $I\subseteq\sqrt[S]{I}$ and $J\subseteq\sqrt[S]{J}$, $I+J\subseteq\sqrt[S]{I}+\sqrt[S]{J}$ and so $\sqrt[S]{I+J}\subseteq\sqrt[S]{\sqrt[S]{I}+\sqrt[S]{J}}$. For the other inclusion, take $a\in\sqrt[S]{\sqrt[S]{I}+\sqrt[S]{J}}$. Then $sa^n\in\sqrt[S]{I}+\sqrt[S]{J}$ for some $s\in S$ and $n\in\mathbb{N}$. So, $sa^n=x+y$ where $x\in\sqrt[S]{I}$ and $y\in\sqrt[S]{J}$. Then there exist $s_1,s_2\in S$ and $k,l\in\mathbb{N}$ such that $s_1x^k\in I$ and $s_2y^l\in J$. This gives $ss_1s_2a\in\sqrt{I+J}$ where $ss_1s_2\in S$ and so $a\in\sqrt[S]{I+J}$. It follows that $\sqrt[S]{\sqrt[S]{I}+\sqrt[S]{J}}\subseteq\sqrt[S]{I+J}$ and so the equality holds.

(4) Since $I\subseteq\sqrt[S]{I}$, $\sqrt[S]{I}\subseteq\sqrt[S]{\sqrt[S]{I}}$. On the other hand, let $a\in\sqrt[S]{\sqrt[S]{I}}$. Then $sa\in\sqrt[S]{\sqrt[S]{I}}=\sqrt[S]{I}$ for some $s\in S$. Hence, $s'sa\in\sqrt{I}$ for some $s'\in S$ and so $a\in\sqrt[S]{I}$ as needed.

(5) If $S^{-1}(\sqrt{I}+\sqrt{J})=S^{-1}R$, then $(\sqrt{I}+\sqrt{J})\cap S\neq\emptyset$. Thus, clearly $(\sqrt[S]{I}+\sqrt[S]{J})\cap S\neq\emptyset$ and $S^{-1}(\sqrt[S]{I}+\sqrt[S]{J})=S^{-1}R$. Conversely, suppose $S^{-1}(\sqrt[S]{I}+\sqrt[S]{J})=S^{-1}R$. Since $(\sqrt[S]{I}+\sqrt[S]{J})\cap S\neq\emptyset$, we can choose $s\in(\sqrt[S]{I}+\sqrt[S]{J})\cap S$. Write $s=a+b$ such that $s_1a\in\sqrt{I}$, $s_2b\in\sqrt{J}$ for some $s_1,s_2\in S$. Then $ss_1s_2\in\sqrt{I}+\sqrt{J}$ and so $(\sqrt{I}+\sqrt{J})\cap S\neq\emptyset$. Hence, $S^{-1}(\sqrt{I}+\sqrt{J})=S^{-1}R$. Similarly, we can obtain that $S^{-1}(\sqrt{I}+\sqrt{J})=S^{-1}R$ iff $S^{-1}(I+J)=S^{-1}R$.

(6) Assume that $\sqrt[S]{I}+\sqrt[S]{J}=R$. Then $S^{-1}(\sqrt[S]{I}+\sqrt[S]{J})=S^{-1}R$ and so by (5), $S^{-1}(I+J)=S^{-1}R$.

(7) This follows since $\left(\sqrt{S^{-1}(IJ)}\right)^c=\left(\sqrt{(S^{-1}I)(S^{-1}J)}\right)^c=\left(\sqrt{(S^{-1}I)\cap(S^{-1}J)}\right)^c$ $=\left(\sqrt{S^{-1}I}\right)^c\cap\left(\sqrt{S^{-1}J}\right)^c$. □

Let S be a multiplicatively closed subset of a ring R. If I is an ideal of R disjoint with S, then we denote the set $\left\{s\in S: sa\in\sqrt{I}\text{ for some }0\neq a\in R\right\}$ by S_I. Recall from [21] that an ideal I of R disjoint with S is called quasi-S-primary if there exists an $s\in S$ such that whenever $a,b\in R$ and $ab\in I$, then $sa\in\sqrt{I}$ or $sb\in\sqrt{I}$.

Proposition 2.6 *Let S be a multiplicatively closed subset of a ring R and I be an ideal of R. If I is quasi S-primary, then $\sqrt[S]{I}$ is prime. The converse part also holds if S_I is finite.*

Proof Since I is quasi S-primary, then there exists $s\in S$ such that whenever $a,b\in R$ such that $ab\in I$, then $sa\in\sqrt{I}$ or $sb\in\sqrt{I}$. Let $a,b\in R$ such that $ab\in\sqrt[S]{I}$ and $a\notin\sqrt[S]{I}$. Then $tab\in\sqrt{I}$ for some $t\in S$. As I is quasi S-primary and $sta\notin\sqrt{I}$, then $sb\in\sqrt{I}$. Thus, $b\in\sqrt[S]{I}$ and $\sqrt[S]{I}$ is prime. Conversely, suppose $\sqrt[S]{I}$ is prime, S_I is finite and $t=\prod_{s\in S_I}s\in S$. Let $ab\in I$ for some $a,b\in R$. Since $ab\in\sqrt[S]{I}$ and $\sqrt[S]{I}$ is prime, $s_1 a \in \sqrt{I}$ or $s_2 b \in \sqrt{I}$ for some $s_1, s_2 \in S$. Thus, $ta \in \sqrt{I}$ or $tb \in \sqrt{I}$ and I is an S-quasi primary ideal. □

Theorem 2.7 *Let $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers. Let S be a multiplicatively closed subset of $\mathbb{Z}_n$ and $I = \langle p_1^{t_1} p_2^{t_2} ... p_m^{t_m} \rangle$ a proper ideal of $\mathbb{Z}_n$ with $I \cap S = \emptyset$ where $1 \leq t_i \leq r_i$ for $i = 1, 2, \cdots, m$. Then $\sqrt[S]{I} = \bigcap\limits_{\substack{i=1 \\ \langle p_i \rangle \cap S = \emptyset}}^{m} \langle p_i \rangle$.*

Proof Let $x \in \sqrt[S]{I}$ and let $i \in \{1, 2, \cdots, m\}$ such that $\langle p_i \rangle \cap S = \emptyset$. Then there exists $s \in S$ such that $sx \in \sqrt{I} = \langle p_1 p_2 ... p_m \rangle \subseteq \langle p_i \rangle$. Since $s \notin \langle p_i \rangle$, then $x \in \langle p_i \rangle$, and so $x \in \bigcap\limits_{\substack{i=1 \\ \langle p_i \rangle \cap S = \emptyset}}^{m} \langle p_i \rangle$. Conversely, let $x \in \bigcap\limits_{\substack{i=1 \\ \langle p_i \rangle \cap S = \emptyset}}^{m} \langle p_i \rangle$. If $\langle p_i \rangle \cap S = \emptyset$ for all $i \in \{1, 2, \cdots, m\}$, then $x \in \langle p_1 p_2 ... p_m \rangle = \sqrt{I} \subseteq \sqrt[S]{I}$. Suppose, say, $\langle p_i \rangle \cap S = \emptyset$ for $i \in \{1, 2, \cdots, t\}$ and $\langle p_i \rangle \cap S \neq \emptyset$ for $i \in \{t+1, \cdots, m\}$. Then clearly, we can find $s \in S \cap \langle p_{t+1} ... p_m \rangle$ and so $sx \in \langle p_1 p_2 ... p_m \rangle = \sqrt{I}$. Therefore, $x \in \sqrt[S]{I}$ and the required equality holds. □

For any prime p dividing n, we denote the multiplicatively closed subset $\{1, p, p^2, p^3, ...\}$ of $\mathbb{Z}_n$ by S_p. As a corollary of Theorem 2.7, we determine the S_p-radical of the ideals of the ring $\mathbb{Z}_n$ for any prime p dividing n.

Corollary 2.8 *Let $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers and let $I = \langle p_1^{t_1} p_2^{t_2} ... p_m^{t_m} \rangle$ a proper ideal of $\mathbb{Z}_n$ with $I \cap S = \emptyset$ where $1 \leq t_i \leq r_i$ for $i = 1, 2, \cdots, m$. Then*

$$\sqrt[S_{p_i}]{I} = \left\{ \begin{array}{ll} \langle p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_k \rangle & i \in \{1, 2, \cdots, m\} \\ \sqrt{I} & i \in \{m+1, m+2, \cdots, k\} \end{array} \right\}$$

For example, in the ring $\mathbb{Z}_{360}$, we have $\sqrt[S_2]{\langle 12 \rangle} = \langle 3 \rangle$, $\sqrt[S_3]{\langle 12 \rangle} = \langle 2 \rangle$ and $\sqrt[S_5]{\langle 12 \rangle} = \langle 6 \rangle = \sqrt{\langle 12 \rangle}$. Note that if $m = 1$ in the previous corollary, then $\sqrt[S_{p_1}]{I} = R$ and $\sqrt[S_{p_i}]{I} = \sqrt{I} = \langle p_1 \rangle$ for all $2 \leq i \leq k$.

Corollary 2.9 *Let $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers. Then $N_{S_{p_i}}(\mathbb{Z}_n) = \sqrt[S_{p_i}]{0} = \langle p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_k \rangle$ for all $i = 1, 2, \cdots, k$.*

Let S be a multiplicatively closed subset of the ring of integers $\mathbb{Z}$. Next, we determine the S-radical of any ideal in $\mathbb{Z}$ disjoint with S.

Theorem 2.10 *Let S be a multiplicatively closed subset of the ring $\mathbb{Z}$ and let $I = \langle n \rangle$ be a proper ideal of of the ring $\mathbb{Z}$ with $I \cap S = \emptyset$ where $n = p_1^{t_1} p_2^{t_2} ... p_m^{t_m}$ for distinct prime integers $p_1, p_2, ..., p_m$. Then $\sqrt[S]{I} = \bigcap\limits_{\substack{i=1 \\ \langle p_i \rangle \cap S = \emptyset}}^{m} \langle p_i \rangle$.*

Proof Similar to the proof of Theorem 2.7. □

Specially, by considering $S_p = \{1, p, p^2, p^3, ...\}$ of $\mathbb{Z}$ where p is a prime integer, we have:

Corollary 2.11 *Let* $I = \langle p_1^{r_1} p_2^{r_2} ... p_k^{r_k} \rangle$ *be a proper ideal of* $\mathbb{Z}$ *and* p *be any prime integer. Then*

$$\sqrt[S_p]{I} = \begin{cases} \langle p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_k \rangle & p = p_i \text{ for some } i \in \{1, 2, \cdots k\} \\ \sqrt{I} & p \notin \{p_1, p_2, \cdots, p_k\} \end{cases}$$

Note that $N_{S_p}(\mathbb{Z}) = N(\mathbb{Z}) = \langle 0 \rangle$ for any prime integer p in $\mathbb{Z}$.

3 Strongly *S*-*n*-ideals

Definition 3.1 Let R be a ring, S be a multiplicatively closed subset of R and I be a proper ideal of R disjoint with S. Then I is called a strongly S -n-ideal if for $a, b \in R$, $ab \in I$ and $a \notin I$ imply $b \in \sqrt[S]{0}$.

For every ideal I of R disjoint with S, clearly, we have the following implications

$$n\text{-ideal} \longrightarrow \text{Strongly } S\text{-}n\text{-ideal} \longrightarrow S\text{-}n\text{-ideal}$$

We show in the next two examples that the arrows above are irreversible.

Example 3.2 (**S-n-ideal that is not strongly S-n-ideal**) Consider the multiplicatively closed subset $S = \{p^n : n \in \mathbb{N}\}$ of the ring $\mathbb{Z}_{pq}$ where p and q are distinct prime integers. Then the zero ideal is S-n-ideal by [12, Theorem 2(2)], but it is not strongly S-n-ideal as $qp \in < 0 >$ and $q \notin < 0 >$ but $p \notin \sqrt[S]{< 0 >} =< q >$.

Example 3.3 (**Strongly S-n-ideal that is not n-ideal**) Let $R = \mathbb{Z}_{12}$ and $S = \{1, 2, 4, 8\}$. Then the ideal $I =< 3 >$ of R is a strongly S-n-ideal which is not an n-ideal. Suppose that $ab \in < 3 >$ and $a \notin < 3 >$. Then $b \in < 3 >= \sqrt[S]{0}$ and so $< 3 >$ is a strongly S -n-ideal. On the other hand, $< 3 > \nsubseteq \sqrt{0} =< 6 >$ and so $< 3 >$ is not an n-ideal.

Now, we give a useful relationship between this class of ideals and the S-nilradical.

Proposition 3.4 *Let* I *be an* S*-prime ideal. Then* I *is strongly* S-n*-ideal if and only if* $(I : s_I) = \sqrt[S]{0}$ *where* s_I *is an* S*-prime element of* I.

Proof Take $a \in (I : s_I)$. Then $s_I a \in I$. As $S \cap I = \emptyset$ and I is strongly S-n-ideal, we get $a \in \sqrt[S]{0}$. For the reverse inclusion, choose $a \in \sqrt[S]{0}$. Then $s' a^n = 0$ for some $s' \in S$ and $n \in \mathbb{Z}^+$. So $a^n \in (0 : s') \subseteq (I : s')$. Then $(I : s') \subseteq (I : s_I)$ and $(I : s_I)$ is a prime ideal by [22, Proposition 2]. Thus $a \in (I : s_I)$, as desired. For the converse, assume that $(I : s_I) = \sqrt[S]{0}$. Let $ab \in I$. Then $s_I a \in I$ or $s_I b \in I$. This gives $a \in (I : s_I) = \sqrt[S]{0}$ or $b \in (I : s_I) = \sqrt[S]{0}$, as needed. □

In the next two theorems, we characterize strongly S-n-ideals of a ring.

Theorem 3.5 *Let R be a ring, S be a multiplicatively closed subset of R and I be an ideal of R with $I \cap S = \emptyset$. Then the following are equivalent.*

1. I is a strongly S-n-ideal of R.
2. Whenever J and K are ideals of R with $JK \subseteq I$ and $J \nsubseteq I$, then $K \subseteq \sqrt[S]{0}$.

Proof (1)⇒(2): Assume I is strongly S-n-ideal of R. Choose $JK \subseteq I$ but $J \nsubseteq I$ and $K \nsubseteq \sqrt[S]{0}$. Then there exist $a \in J$, $b \in K$ such that $ab \in I$ but $a \notin I$ and $b \notin \sqrt[S]{0}$, a contradiction.

(2)⇒(1): Let $ab \in I$ with $a \notin I$. Put $J = (a)$ and $K = (b)$. Then $ab \in (a)(b) = (ab) \subseteq I$ with $(a) \nsubseteq I$ and this implies $b \in (b) \subseteq \sqrt[S]{0}$ which completes the proof. □

Theorem 3.6 *Let R be a ring, S be a multiplicatively closed subset of R, I be an ideal of R with $I \cap S = \emptyset$ and suppose $\sqrt[S]{0}$ is strongly S -n-ideal of R. Then the following are equivalent.*

1. I is a strongly S-n-ideal of R.
2. For ideals $I_1, I_2, \cdots I_n$ of R, $I_1I_2 \ldots I_n \subseteq I$ implies either $I_k \subseteq I$ or $I_l \subseteq \sqrt[S]{0}$ for some $k, l \in \{1, 2, \cdots, n\}$.
3. For elements $a_1, a_2 \cdots, a_n$ of R, $a_1a_2 \ldots a_n \in I$ implies either $a_k \in I$ or $a_l \in \sqrt[S]{0}$ for some $k, l \in \{1, 2, \cdots, n\}$.

Proof (1)⇒(2): Apply induction on n. If $n = 2$, we are done by Theorem 3.5. Suppose $n \geq 3$ and the statement holds for $n - 1$. Let $I_1, I_2, \cdots I_n$ be ideals of R with $I_1I_2 \ldots I_n \subseteq I$. Then by Theorem 3.5, $I_1I_2 \ldots I_{n-1} \subseteq I$ or $I_n \subseteq \sqrt[S]{0}$. Assume that $I_1I_2 \ldots I_{n-1} \subseteq I$. Then we have either $I_1I_2 \ldots I_{n-2} \subseteq I$ or $I_{n-1} \subseteq \sqrt[S]{0}$. By continuing this way, we obtain the desired result.

(2)⇒(3): Put $I_k = (a_k)$.

(3)⇒(1): It is clear by taking $n = 2$. □

Proposition 3.7 *Let R be a ring, S be a multiplicatively closed subset of R and I be an ideal of R with $I \cap S = \emptyset$. Then we have the following.*

1. *If I is strongly S-n-ideal, then $I \subseteq \sqrt[S]{0}$. In particular, if $S \subseteq reg(R)$, then $I \subseteq \sqrt{0}$.*
2. *$\sqrt[S]{0}$ is a strongly S-n-ideal if and only if $\sqrt[S]{0}$ is prime.*

Proof (1) Let $a \cdot 1 = a \in I$. Since $1 \notin I$ and I is strongly S -n-ideal, we have $a \in \sqrt[S]{0}$, as needed. If $S \subseteq reg(R)$, $sa^n = 0$ implies $a \in \sqrt{0}$.

(2) It is straightforward. □

Note that the converse of Proposition 3.7(1) is not true. For instance, consider the multiplicatively closed subset $S = \{1, 2, 4\}$ and the ideal $I =< 0 >$ of $\mathbb{Z}_6$. Then $I \subseteq \sqrt[S]{0} =< 3 >$, but I is not strongly S-n-ideal by Example 3.2. Moreover, in the "particular" part, the condition $S \subseteq reg(R)$ is necessary. For instance, consider the multiplicatively closed subset $S = \{p^n : n \in \mathbb{N}\}$ of the ring $\mathbb{Z}_{p^kq^t}$ where p, q are distinct prime integers and $k, t \geq 1$. Then the ideal $J =< q >$ is prime since $J = \sqrt[S]{0}$ is strongly S-n-ideal by Proposition 3.7(2). Here, since $S \nsubseteq reg(\mathbb{Z}_{p^kq^t})$ observe that $J \nsubseteq \sqrt{0} =< pq >$.

We recall that a ring R is said to be a UN-ring if every nonunit element is a product of a unit and a nilpotent. In the following, we present a new characterization for UN-rings in terms of strongly S-n-ideals.

Proposition 3.8 *For an m.c.s $S \subseteq reg(R)$ of a ring R, the following statements are equivalent.*

1. Every proper ideal of R is an n-ideal.
2. Every proper ideal of R is a strongly S-n-ideal.
3. R is a UN-ring.

Proof (1)⇒(2) Clear.

(2)⇒(3) . We show that R has a unique prime ideal. Suppose that P is a prime ideal of R. Then, from our assumption, P is a strongly S-n-ideal, hence $P \cap S = \emptyset$, P is also an S-prime ideal. So, P is an S-prime and an S-n-ideal by our assumption (2). Hence, $P \subseteq (P : s) = \sqrt[S]{0}$ by Proposition 3.4. Since $S \subseteq reg(R)$, we have $P \subseteq \sqrt[S]{0} = \sqrt{0}$, and thus $P = \sqrt{0}$ is the only prime ideal of R. Therefore, R is a UN-ring by [6, Proposition 2 (3)].

(3)⇒(1) Follows from [19, Proposition 2.25]. □

Let $n \in \mathbb{N}$ and let $p < n$ be a prime integer. Next, we determine all S_p-n-ideals of the ring $\mathbb{Z}_n$.

Theorem 3.9 *Let $n = p_1^{r_1} p_2^{r_2} ... p_k^{r_k}$ where $p_1, p_2, ..., p_k$ are distinct prime integers and $r_i \geq 1$ for $i = 1, 2, \cdots, k$. Then*

1. If for $i = 1, 2, \cdots, k$, $\mathbb{Z}_n$ has a strongly S_{p_i}-n-ideal, then $k = 2$.
2. If $n = p_1^{r_1} p_2^{r_2}$, then the only strongly S_{p_1}-n-ideal of $\mathbb{Z}_n$ are $\left\langle p_2^{t_2} \right\rangle$ where $1 \leq t_2 \leq r_2$ and the only strongly S_{p_2}-n-ideal of $\mathbb{Z}_n$ are $\left\langle p_1^{t_1} \right\rangle$ where $1 \leq t_1 \leq r_1$.
3. Let $q < n$ be a prime integer distinct from $p_i' s$ for $i = 1, 2, ..., k$. Then $\mathbb{Z}_n$ has a strongly S_q-n-ideal if and only if $k = 1$. Furthermore, every proper ideal of $\mathbb{Z}_{p_i^{r_i}}$ is a strongly S_q-n-ideal.

Proof (1) Suppose $k = 1$ ($n = p_1^{r_1}$). Then $\mathbb{Z}_n$ has no strongly S_{p_1}-n-ideals as $I \cap S_{p_1} \neq \emptyset$ for any ideal I of $\mathbb{Z}_n$. Suppose $k \geq 3$ and note that $\sqrt[S_{p_1}]{0} = \langle p_2 \cdots p_k \rangle$ by Corollary 2.9. If I is a strongly S_{p_1}-n-ideal of $\mathbb{Z}_n$, then $I \subseteq \sqrt[S_{p_1}]{0} = \langle p_2 \cdots p_k \rangle$ by Proposition 3.7(i). Assume $I = \left\langle p_2^{t_2} \cdots p_k^{t_k} \right\rangle$ where $t_i \leq r_i$ for all $i = 2, 3, \cdots .k$. Then, $p_2^{t_2}(p_3^{t_3} ... p_k^{t_k}) \in I$ but $p_2^{t_2} \notin I$ and $p_3^{t_3} ... p_k^{t_k} \notin \sqrt[S_{p_1}]{0}$ which is a contradiction. Thus, I is not a strongly S_{p_1}-n-ideal of $\mathbb{Z}_n$. In general, there is no strongly S_{p_i}-n-ideal of $\mathbb{Z}_n$ for all $i = 1, 2, ..., k$. Therefore, we must have $k = 2$.

(2) Suppose $1 \leq t_2 \leq r_2$ and note that $\left\langle p_2^{t_2} \right\rangle \cap S_{p_1} = \emptyset$. If $ab \in \left\langle p_2^{t_2} \right\rangle$ and $a \notin \sqrt[S_{p_1}]{0} = \langle p_2 \rangle = \left\langle p_2^{t_2} \right\rangle$, then $b \in \left\langle p_2^{t_2} \right\rangle$ as $\left\langle p_2^{t_2} \right\rangle$ is primary in $\mathbb{Z}_n$. Thus, $\left\langle p_2^{t_2} \right\rangle$ is a strongly S_{p_1}-n-ideal of $\mathbb{Z}_n$. Similarly, $\left\langle p_1^{t_1} \right\rangle$ is a strongly S_{p_2}-n-ideal of $\mathbb{Z}_n$. Now, suppose $I = \left\langle p_1^{t_1} p_2^{t_2} \right\rangle$ where $1 \leq t_1, t_2 \leq r_2$. Then $p_1^{t_1} p_2^{t_2} \in I$ but $p_1^{t_1} \notin \sqrt[S_{p_1}]{0} = \langle p_2 \rangle$ and $p_2^{t_2} \notin I$. Thus, I is not a strongly S_{p_1}-n-ideal of $\mathbb{Z}_n$. Similarly, I is not a strongly S_{p_2}-n-ideal of $\mathbb{Z}_n$.

(3) Let $I = \left\langle p_1^{t_1} p_2^{t_2} \cdots p_k^{t_k} \right\rangle$ where $t_i \leq r_i$ for all $i = 1, 2, \cdots k$ be a proper ideal of $\mathbb{Z}_n$. It is clear that $I \cap S_q = \emptyset$. From Theorem 2.7, we have $\sqrt[S_q]{0} = \sqrt{0} = \langle p_1 \cdots p_k \rangle$. Assume that $k \geq 2$ and I is a strongly S-n-ideal of $\mathbb{Z}_n$. Then, we have $p_1^{t_1}(p_2^{t_2} ... p_k^{t_k}) \in I$ but $p_1^{t_1} \notin I$ and $p_2^{t_2} ... p_k^{t_k} \notin \sqrt[S_q]{0}$, a contradiction. Hence, I is not a strongly S_q-n-ideal of $\mathbb{Z}_n$. Thus, if $\mathbb{Z}_n$ has a strongly S_q-n-ideal, then $k = 1$. Now, suppose that $k = 1$. Since $S_q \subseteq reg(\mathbb{Z}_{p_i^{r_i}})$, every proper ideal of $\mathbb{Z}_{p_i^{r_i}}$ is a strongly S_q-n-ideal by Proposition 3.8. □

The following lemma will be useful in the proof of Proposition 3.11.

Lemma 3.10 *Let $f : R_1 \to R_2$ be ring homomorphism and S be a multiplicatively closed subset of R_1. Then $f(\sqrt[S]{0_{R_1}}) \subseteq \sqrt[f(S)]{0_{R_2}}$. If moreover, f is an epimorphism and $Kerf \subseteq \sqrt[S]{0_{R_1}}$, then the equality holds.*

Proof Let $b = f(a) \in f(\sqrt[S]{0_{R_1}})$ where $a \in \sqrt[S]{0_{R_1}}$. Then $sa \in \sqrt{0_{R_1}}$ for some $s \in S$ and so $f(s)b^m = f(sa^m) = 0_{R_2}$ for some $m \in \mathbb{N}$. Thus, $b \in \sqrt[f(S)]{0_{R_2}}$ and $f(\sqrt[S]{0_{R_1}}) \subseteq \sqrt[f(S)]{0_{R_2}}$. Conversely, suppose f is an epimorphism and $Kerf \subseteq \sqrt[S]{0_{R_1}}$. Let $b = f(a) \in \sqrt[f(S)]{0_{R_2}}$ and choose $s \in S$, $m \in \mathbb{N}$ such that $f(sa^m) = f(s)b^m = 0_{R_2}$. Then $sa^m \in Kerf \subseteq \sqrt[S]{0_{R_1}}$ and so $a \in \sqrt[S]{\sqrt[S]{0_{R_1}}} = \sqrt[S]{0_{R_1}}$. Therefore, $b = f(a) \in f(\sqrt[S]{0_{R_1}})$ and the other containment holds. □

Note that even if $Kerf \subseteq \sqrt[S]{0_{R_1}}$, the reverse inclusion in Lemma 3.10 need not be true if f is not an epimorphism. For example, consider $R = \mathbb{Z}_{12}$ and $S = \{1, 2, 4, 8\}$ and $f : R \to R$ defined by $f(x) = 4x$. Then $f(S) = \{1, 4, 8\}$ and $Kerf = \sqrt[S]{0_{R_1}} = \sqrt[f(S)]{0_{R_2}} =< 3 >$ but, $f(\sqrt[S]{0_{R_1}}) = 0_{R_2}$.

Now, we are ready to give the following result.

Proposition 3.11 *Let $f : R_1 \to R_2$ be a ring epimorphism and S be a multiplicatively closed subset of R_1. Then the following statements hold.*

1. If I is a strongly S-n-ideal of R_1 such that $Kerf \subseteq I$, then $f(I)$ is a strongly $f(S)$-n-ideal of R_2.
2. If J is a strongly $f(S)$-n-ideal of R_2 and $Kerf \subseteq \sqrt[S]{0_{R_1}}$, then $f^{-1}(J)$ is a strongly S-n-ideal of R_1.

Proof Firstly note that $f(I) \cap f(S) = \emptyset$.

(1) Let $xy \in f(I)$ with $x \notin f(I)$ for $x, y \in R_2$. Then there exist $a, b \in R_1$ such that $f(a) = x$ and $f(b) = y$. So $f(a)f(b) = f(ab) \in f(I)$. This implies that $ab \in f^{-1}(f(I)) = I + Kerf = I$. If $a \in I$ then $x = f(a) \in f(I)$, a contradiction. Thus $ab \in I$ with $a \notin I$. Then we conclude that $b \in \sqrt[S]{0_{R_1}}$ giving $y = f(b) \in f(\sqrt[S]{0_{R_1}}) \subseteq \sqrt[f(S)]{0_{R_2}}$.

(2) Take $ab \in f^{-1}(J)$. Then $f(a)f(b) = f(ab) \in J$. As J is a strongly $f(S)$-n-ideal of R_2, $f(a) \in J$ or $f(b) \in \sqrt[f(S)]{0_{R_2}} = f(\sqrt[S]{0_{R_1}})$. Thus we obtain $a \in f^{-1}(J)$ or $b \in f^{-1}(f(\sqrt[S]{0_{R_1}})) = \sqrt[S]{0_{R_1}} + Kerf = \sqrt[S]{0_{R_1}}$, as desired. □

In view of Proposition 3.11, we have the following result.

Corollary 3.12 *Let R be a ring, S be a multiplicatively closed subset and I, J be ideals of R with $I \subseteq J$. Then we have the following.*

1. If J is a strongly S-n-ideal of R, then J/I is a strongly $\overline{S}$-n-ideal of R/I where $\overline{S} = \{s + I : s \in S\}$. The converse is true if $I \subseteq \sqrt[S]{0}$.
2. If $R \leq R'$ and I' is a strongly S-n-ideal of R', then $I' \cap R$ is a strongly S-n-ideal of R.

Proof (1) Suppose $a+I \in (J/I)\cap\overline{S}$. Then $a \in J\cap S$, a contradiction. So $(J/I)\cap\overline{S} = \emptyset$. Now consider the natural map $\pi : R \to R/I$. Since J is strongly S-n-ideal, $\pi(J) = J/I$ is a strongly $\overline{S}$-n-ideal of R/I by Proposition 3.11 (i).

(2) It is clear that $(I'\cap R)\cap S = \emptyset$. Let $ab \in I'\cap R$. Since $ab \in I'$ and I' is a strongly S-n-ideal of R', $a \in I'$ or $b \in \sqrt[S]{0_{R'}}$. They give $a \in I'\cap R$ or $b \in \sqrt[S]{0_{R'}} \subseteq \sqrt[S]{0_R}$, as needed. □

Proposition 3.13 *Let R be a ring, S be a multiplicatively closed subset of R, I be an ideal of R with $I \cap S = \emptyset$. Then $S^{-1}I$ is an n-ideal of $S^{-1}R$ and $S^{-1}I \cap R = I$ if and only if I is a strongly S-n-ideal of R.*

Proof Suppose that $S^{-1}I$ is an n-ideal of $S^{-1}R$ and $S^{-1}I \cap R = I$. Choose $ab \in I$ with $a \notin I$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}I$ with $\frac{a}{1} \notin S^{-1}I$. If $\frac{a}{1} \in S^{-1}I$, $ta \in I$ for some $t \in S$. Then $a = \frac{ta}{t} \in S^{-1}I \cap R = I$, a contradiction. This implies that $\frac{b}{1} \in \sqrt{S^{-1}0}$. Then there exists $n \in \mathbb{Z}^+$ such that $(\frac{b}{1})^n \in S^{-1}0$. So $ub^n = 0$ for some $u \in S$. This gives $b \in \sqrt[S]{0}$, as needed. Conversely, assume that I is a strongly S-n-ideal of R and let $\frac{a}{s_1}\frac{b}{s_2} \in S^{-1}I$ with $\frac{a}{s_1} \notin S^{-1}I$. Then $uab \in I$ for some $u \in S$ which implies that either $ua \in I$ or $b \in \sqrt[S]{0}$. In the former case, we have $\frac{a}{s_1} = \frac{ua}{us_1} \in S^{-1}I$ which gives a contradiction. If the latter case holds, then there is an $s \in S$ such that $sb \in \sqrt{0}$. Thus, $\frac{b}{s_2} = \frac{sb}{ss_2} \in S^{-1}\sqrt{0} \subseteq \sqrt{S^{-1}0}$ and so $S^{-1}I$ is an n-ideal of $S^{-1}R$. Now we will show that $S^{-1}I \cap R = I$. It is clear that $I \subseteq S^{-1}I \cap R$. For the reverse inclusion, choose $r \in S^{-1}I \cap R$. Then $\frac{r}{1} = \frac{i}{s}$ for some $i \in I$ and $s \in S$. So there exists $u \in S$ such that $ur \in I$. Since I is strongly S-n-ideal, $r \in I$ or $u \in \sqrt[S]{0}$. If $u \in \sqrt[S]{0}$, $s'u^n = 0 \in S$ for some $s' \in S$ which is a contradiction. Hence we conclude that $r \in I$. □

Remark 3.14 Let R, R' be rings, S, S' be multiplicatively closed subset and I, J be proper ideal of R, R' disjoint with S, S', respectively. Then $I \times J$ is not a strongly $S \times S'$-n-ideal of $R \times R'$. Indeed, if $I \times J$ is a strongly $S \times S'$-n-ideal of $R \times R'$, then since $(1, 0)(0, 1) \in I \times J$, we have either $(1, 0) \in I \times J$ or $(0, 1) \in \sqrt[S\times S']{0_{R\times R'}}$. If $(1, 0) \in I \times J$, then we get $1 \in I$, a contradiction. Suppose $(0, 1) \in \sqrt[S\times S']{0_{R\times R'}}$. This means that $(s, s')(0, 1)^n = (0, 0)$ for some $(s, s') \in S \times S'$ and $n \in \mathbb{Z}^+$. Thus $s' = 0$ which gives again a contradiction.

Let M be an R-module. We recall that $R \propto M = \{(r, m) : r \in R, m \in M\}$ with component-wise addition and multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity $(1, 0)$ and it is called the idealization of M. For an ideal I of R and a submodule N of M, $I \propto N$ is an ideal of $R \propto M$ if and only if $IM \subseteq N$. Now, we will investigate the relationship between strongly S-n-ideals of R and strongly $S \propto M$-n-ideals of $R \propto M$. First, we need the following lemma.

Lemma 3.15 *Let M be an R-module, N be a submodule of M and I an ideal of a ring R such that $IM \subseteq N$. Then $\sqrt[S\propto M]{I \propto N} = \sqrt[S]{I} \propto M$.*

Proof Let $(a, m) \in \sqrt[S\propto M]{I \propto N}$. Then there exist $(s, m') \in S \propto M$ and $n \in \mathbb{Z}^+$ such that $(s, m')(a, m)^n = (s, m')(a^n, na^{n-1}m) \in I \propto N$. So $sa^n \in I$ implying $a \in \sqrt[S]{I}$. Thus $(a, m) \in \sqrt[S]{I} \propto M$ and $\sqrt[S\propto M]{I \propto N} \subseteq \sqrt[S]{I} \propto M$. For the reverse inclusion, let $(a, m) \in \sqrt[S]{I} \propto M$. Then $sa^n \in I$ for some $s \in S$. $(s, sm')(a, m)^{n+1} = (sa^{n+1}, s(n+1)a^n m + sm'a^{n+1}) \in I \propto IM \subseteq I \propto N$. Thus $(a, m) \in \sqrt[S\propto M]{I \propto N}$, which completes the proof. □

Proposition 3.16 *Let M be an R-module and N be a submodule of M with $IM \subseteq N$. If $I \propto N$ is a strongly $S \propto M$-n-ideal of $R \propto M$, then I is a strongly S-n-ideal of R.*

Proof First note that since $(I \propto N) \cap (S \propto M) = (I \cap S) \propto N = \emptyset$, $I \cap S = \emptyset$. Now choose $ab \in I$. Then $(a, 0)(b, 0) \in I \propto N$. As $I \propto N$ is a strongly $S \propto M$-n-ideal of $R \propto M$, $(a, 0) \in I \propto N$ or $(b, 0) \in \sqrt[S\propto M]{0 \propto M} = \sqrt[S]{0} \propto M$. This gives $a \in I$ or $b \in \sqrt[S]{0}$, as desired. □

The converse of the previous proposition may not be true as we can see in the following example.

Example 3.17 Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{15}$, $S = \{-1, 1\}$ and $N = \overline{0}$. Here, (0) is a strongly S-n-ideal. Although $(3, 0)(0, 5) \in 0 \propto \overline{0}$, neither $(0, 5) \in 0 \propto \overline{0}$ nor $(3, 0) \in \sqrt[S\propto M]{0 \propto \overline{0}} = \sqrt[S]{0} \propto M$.

Proposition 3.18 *Let M be an R-module and I be an ideal of a ring R. If I is strongly S-n-ideal of R, then $I \propto M$ is a strongly $S \propto M$-n-ideal of $R \propto M$.*

Proof Let $(a, m)(a, m') \in I \propto M$. Then $aa' \in I$ and this implies that $a \in I$ or $a' \in \sqrt[S]{0}$. Thus $(a, m) \in I \propto M$ or $(a', m') \in \sqrt[S]{0} \propto M = \sqrt[S\propto M]{0 \propto M}$. □

4 Strongly *S*-*n*-ideals in amalgamated algebra

Let R and R' be two rings, J be an ideal of R' and $f : R \to R'$ be a ring homomorphism. The set $R \bowtie^f J = \{(r, f(r) + j) : r \in R, j \in J\}$ is a subring of $R \times R'$ (with identity element $(1_R, 1_{R'})$) called the amalgamation of R and R' along J with respect to f. In particular, if $Id_R : R \to R$ is the identity homomorphism on R, then $R \bowtie J = R \bowtie^{Id_R} J = \{(r, r + j) : r \in R, j \in J\}$ is the amalgamated duplication of a ring along an ideal J. This construction has been first defined and studied by D'Anna and Fontana, [7]. Many properties of this ring have been investigated and analyzed over the last two decades, see for example [8], [9].

Let I be an ideal of R and K be an ideal of $f(R) + J$. Then $I \bowtie^f J = \{(i, f(i) + j) : i \in I, j \in J\}$ and $\bar{K}^f = \{(a, f(a) + j) : a \in R, j \in J, f(a) + j \in$

$K\}$ are ideals of $R\bowtie^f J$, [9]. For a multiplicatively closed subset S of R, one can easily verify that $S\bowtie^f J=\{(s,f(s)+j): s\in S,\ j\in J\}$ and $W=\{(s,f(s)): s\in S\}$ are multiplicatively closed subsets of $R\bowtie^f J$.

If I is an ideal of R, then one can easily prove that $\sqrt{I\bowtie^f J}=\sqrt{I}\bowtie^f J$. Moreover, $N(R\bowtie^f J)=\sqrt{0_{R\bowtie^f J}}\subseteq\sqrt{0_R}\bowtie^f J$ and the equality holds if $J\subseteq\sqrt{0_{R'}}$. Indeed, let $(a,f(a)+j)\in\sqrt{0_{R\bowtie^f J}}$ and choose $n\in\mathbb{N}$ such that $(a^n,(f(a)+j)^n)=(a,f(a)+j)^n=0_{R\bowtie^f J}$. Then $a^n=0_R$ and so $(a,f(a)+j)\in\sqrt{0_R}\bowtie^f J$. Suppose moreover that $J\subseteq\sqrt{0_{R'}}$ and let $(a,f(a)+j)\in\sqrt{0_R}\bowtie^f J$. Then there is $m\in\mathbb{N}$ such that $a^m=0_R$ and so $(a,f(a)+j)^m=(a^m,(f(a)+j)^m)=(0_R,j')$ for some $j'\in J$. Now, $J\subseteq\sqrt{0_{R'}}$ implies $j'^k=0_{R'}$ for some $k\in\mathbb{N}$ and so $(a,f(a)+j)^{mk}=0_{R\bowtie^f J}$. Therefore, $(a,f(a)+j)\in\sqrt{0_{R\bowtie^f J}}$ and the equality $\sqrt{0_{R\bowtie^f J}}=\sqrt{0_R}\bowtie^f J$ holds.

Analogous to these facts, we have the similar identities for S-radicals.

Lemma 4.1 *Consider the amalgamation ring* $R\bowtie^f J$ *as above. Let* S *be a multiplicatively closed subset of* R *and* I *be an ideal of* R *disjoint with* S. *Then* $\sqrt[S\bowtie^f J]{I\bowtie^f J}=\sqrt[W]{I\bowtie^f J}=\sqrt[S]{I}\bowtie^f J$ *and* $\sqrt[W]{0_{R\bowtie^f J}}\subseteq\sqrt[S\bowtie^f J]{0_{R\bowtie^f J}}\subseteq\sqrt[S]{0_R}\bowtie^f J$. *If* $J\subseteq\sqrt{0_{R'}}$, *then* $\sqrt[S\bowtie^f J]{0_{R\bowtie^f J}}=\sqrt[W]{0_{R\bowtie^f J}}=\sqrt[S]{0_R}\bowtie^f J$.

Proof Let $(r,f(r)+j)\in\sqrt[S\bowtie^f J]{I\bowtie^f J}$. Then $(s,f(s)+j')(r,f(r)+j)\in\sqrt{I\bowtie^f J}=\sqrt{I}\bowtie^f J$ for some $(s,f(s)+j')\in S\bowtie^f J$. Thus, $sr\in\sqrt{I}$ and $r\in\sqrt[S]{I}$. It follows that $(r,f(r)+j)\in\sqrt[S]{I}\bowtie^f J$ and so $\sqrt[S\bowtie^f J]{I\bowtie^f J}\subseteq\sqrt[S]{I}\bowtie^f J$. The proof of the reverse inclusion is similar. Now, suppose $(r,f(r)+j)\in\sqrt[S\bowtie^f J]{0_{R\bowtie^f J}}$. Then $(s,f(s)+j')(r,f(r)+j)\in\sqrt{0_{R\bowtie^f J}}\subseteq\sqrt{0_R}\bowtie^f J$ for some $(s,f(s)+j')\in S\bowtie^f J$. Thus, $sr\in\sqrt{0_R}$, $r\in\sqrt[S]{0_R}$ and $(r,f(r)+j)\in\sqrt[S]{0_R}\bowtie^f J$. Since $W\subseteq S\bowtie^f J$, we have $\sqrt[W]{0_{R\bowtie^f J}}\subseteq\sqrt[S\bowtie^f J]{0_{R\bowtie^f J}}\subseteq\sqrt[S]{0_R}\bowtie^f J$. If $J\subseteq\sqrt{0_{R'}}$, then the reverse inclusions hold since $\sqrt{0_{R\bowtie^f J}}=\sqrt{0_R}\bowtie^f J$. ☐

Next, for a multiplicatively closed subset S of R, we determine when the ideal $I\bowtie^f J$ is a strongly $(S\bowtie^f J)$-n-ideal and strongly W-n-ideal in $R\bowtie^f J$.

Theorem 4.2 *Consider the amalgamation ring* $R\bowtie^f J$ *as above. Let* S *be a multiplicatively closed subset of* R *and* I *be an ideal of* R *disjoint with* S. *Consider the following statements:*

1. $I\bowtie^f J$ is a strongly W-n-ideal of $R\bowtie^f J$.
2. $I\bowtie^f J$ is a strongly $(S\bowtie^f J)$-n-ideal of $R\bowtie^f J$.
3. I is a strongly S-n-ideal of R.

Then (1) ⇒ (2) ⇒ (3). Moreover, if $J\subseteq\sqrt{0_{R'}}$, then the statements are equivalent.

Proof (1)⇒(2) Clear, as $\sqrt[W]{0_{R\bowtie^f J}}\subseteq\sqrt[S\bowtie^f J]{0_{R\bowtie^f J}}$ by Lemma 4.1.

(2)⇒(3) First note that $(S\bowtie^f J)\cap(I\bowtie^f J)=\emptyset$ if and only if $S\cap I=\emptyset$. Suppose $I\bowtie^f J$ is a strongly $(S\bowtie^f J)$-n-ideal of $R\bowtie^f J$. Let $a,b\in R$ such that $ab\in I$ and $a\notin\sqrt[S]{0_R}$. Then $(a,f(a))(b,f(b))\in I\bowtie^f J$ and $(a,f(a))\notin\sqrt[S]{0_R}\bowtie^f J\supseteq\sqrt[S\bowtie^f J]{0_{R\bowtie^f J}}$. By the assumption, $(b,f(b))\in I\bowtie^f J$ and so $b\in I$ as needed.

Now suppose $J \subseteq \sqrt{0_{R'}}$. We prove (3)⇒(1). Suppose I is a strongly S-n-ideal of R. Let $(a, f(a)+j_1), (b, f(b)+j_2) \in R\bowtie^f J$ such that $(a, f(a)+j_1)(b, f(b)+j_2) = (ab, (f(a)+j_1)(f(b)+j_2)) \in I\bowtie^f J$ and $(a, f(a)+j_1) \notin \sqrt[W]{0_{R\bowtie^f J}} = \sqrt[S]{0_R}\bowtie^f J$. Then $a \notin \sqrt[S]{0_R}$ and since $ab \in I$, we conclude by the assumption that $b \in I$. Thus, $(b, f(b)+j_2) \in I\bowtie^f J$ and $I\bowtie^f J$ is a strongly W-n-ideal of $R\bowtie^f J$. □

Corollary 4.3 *Consider the amalgamation ring $R\bowtie^f J$ as above where $J \subseteq \sqrt{0_{R'}}$. Let S be a multiplicatively closed subset of R. The strongly $(S\bowtie^f J)$-n-ideals of $R\bowtie^f J$ containing $\{0\}\times J$ are of the form $I\bowtie^f J$ where I is a strongly S –n-ideal of R.*

Proof By Theorem 4.2, $I\bowtie^f J$ is a strongly $(S\bowtie^f J)$-n-ideal of $R\bowtie^f J$ for any strongly S-n-ideal I of R. Let K be a strongly $(S\bowtie^f J)$-n-ideal of $R\bowtie^f J$ containing $\{0\}\times J$. Consider the epimorphism $\varphi : R\bowtie^f J \to R$ defined by $\varphi(a, f(a)+j) = a$ for all $(a, f(a)+j) \in R\bowtie^f J$. Since $Ker(\varphi) = \{0\}\times J \subseteq K$, then $I := \varphi(K)$ is a strongly S-n-ideal of R by Proposition 3.11. Since $\{0\}\times J \subseteq K$, we conclude that $K = I\bowtie^f J$. □

Consider the amalgamation ring $R\bowtie^f J$ as above and let K be an ideal of R'. Then by simple computations, we can conclude that $\sqrt{\bar{K}^f} = \overline{\sqrt{K}}^{f}$. Let T be a multiplicatively closed subset of R'. Then clearly, the set $\bar{T}^f = \{(s, f(s)+j) : s \in R,\ j \in J,\ f(s)+j \in T\}$ is a multiplicatively closed subset of $R\bowtie^f J$.

Lemma 4.4 *Consider the amalgamation ring $R\bowtie^f J$ as above and let $T \subseteq f(R)$ be a multiplicatively closed subset of R'. Then*

1. $\sqrt[\bar{T}^f]{0_{R\bowtie^f J}} \subseteq \overline{\sqrt[T]{0_{R'}}}^{f}$ and the equality holds if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$.
2. If K is an ideal of R', then $\sqrt[\bar{T}^f]{\bar{K}^f} = \overline{\sqrt[T]{K}}^{f}$.

Proof (1) Let $(r, f(r)+j) \in \sqrt[\bar{T}^f]{0_{R\bowtie^f J}}$. Then $(s, f(s)+j)\,(r, f(r)+j) \in \sqrt{0_{R\bowtie^f J}}$ for some $(s, f(s)+j) \in \bar{T}^f$. Thus, $(f(s)+j)\,(f(r)+j) \in \sqrt{0_{R'}}$ where $(f(s)+j) \in T$ and so $(f(r)+j) \in \sqrt[T]{0_{R'}}$. Hence, $(r, f(r)+j) \in \overline{\sqrt[T]{0_{R'}}}^{f}$, as required. Now, suppose $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$ and let $(r, f(r)+j) \in \overline{\sqrt[T]{0_{R'}}}^{f}$. Then, $(f(r)+j) \in \sqrt[T]{0_{R'}}$ and so $t(f(r)+j) \in \sqrt{0_{R'}}$ for some $t = f(s) \in T$ as $T \subseteq f(R)$. Since $J \subseteq \sqrt{0_{R'}}$, then $f(sr) = f(s)f(r) \in \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$ implies $sr \in \sqrt{0_R}$. Hence, $(s, t)(r, f(r)+j) \in \sqrt{0_R}\times\sqrt{0_{R'}} \subseteq \sqrt{0_{R\bowtie^f J}}$. Therefore, $(r, f(r)+j) \in \sqrt[\bar{T}^f]{0_{R\bowtie^f J}}$ and the equality holds.

(2) Suppose $T \subseteq f(R)$ and let $(r, f(r)+j) \in \sqrt[\bar{T}^f]{\bar{K}^f}$. Then $(s, f(s)+j)\,(r, f(r)+j) \in \sqrt{\bar{K}^f} = \overline{\sqrt{K}}^{f}$ for some $(s, f(s)+j) \in \bar{T}^f$. Thus, $(f(s)+j')\,(f(r)+j) \in \sqrt{K}$ where $f(s)+j' \in T$ and so $(f(r)+j) \in \sqrt[T]{K}$. It follows that $(r, f(r)+j) \in \overline{\sqrt[T]{K}}^{f}$ and $\sqrt[\bar{T}^f]{\bar{K}^f} \subseteq \overline{\sqrt[T]{K}}^{f}$. Conversely, let $(r, f(r)+j) \in \overline{\sqrt[T]{K}}^{f}$ so that $f(r)+j \in \sqrt[T]{K}$. Then $t(f(r)+j) \in \sqrt{K}$ for some $t \in T$ and so $t^m(f(r)+j)^m \in K$ for some $m \in \mathbb{N}$. Choose $s \in R$ such that $t = f(s)$. Then, $(s, t)^m(r, f(r)+j))^m \in \bar{K}^f$ and so $(s, t)(r, f(r)+j)) \in \sqrt{\bar{K}^f}$ where $(s, t) \in \bar{T}^f$. Therefore, $(r, f(r)+j)) \in \sqrt[\bar{T}^f]{\bar{K}^f}$ and $\overline{\sqrt[T]{K}}^{f} \subseteq \sqrt[\bar{T}^f]{\bar{K}^f}$. □

Theorem 4.5 *Consider the amalgamation ring $R\bowtie^f J$ as above where f is an epimorphism. Let K be an ideal of R' and T be a multiplicatively closed subset of R' disjoint with K. If $\bar{K}^f$ is a strongly $\bar{T}^f$-n-ideal of $R\bowtie^f J$, then K is a strongly T -n-ideal of R'. The converse is true if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$.*

Proof First, note that $T \cap K = \emptyset$ if and only if $\bar{T}^f \cap \bar{K}^f = \emptyset$. Suppose $\bar{K}^f$ is a strongly $\bar{T}^f$ -n-ideal of $R\bowtie^f J$. Let a',$b' \in R'$ such that $a'b' \in K$ and choose $a, b \in R$ where $f(a) = a'$ and $b = f(b')$. Then $(a, f(a)), (b, f(b)) \in R\bowtie^f J$ with $(a, f(a))(b, f(b)) = (ab, f(ab)) \in \bar{K}^f$. By the assumption, we have either $(a, f(a)) \in \sqrt[\bar{T}^f]{0_{R\bowtie^f J}} \subseteq \overline{\sqrt[T]{0_{R'}}}^{f}$ or $(b, f(b)) \in \bar{K}^f$. Thus, $a' = f(a) \in \sqrt[T]{0_{R'}}$ or $b' = f(b) \in K$ and K is a strongly T-n-ideal of R'. Now, suppose K is a strongly T -n-ideal of R', $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$. Let $(a, f(a) + j_1)(b, f(b) + j_2) = (ab, (f(a)+j_1)(f(b)+j_2)) \in \bar{K}^f$ for $(a, f(a)+j_1), (b, f(b)+j_2) \in R\bowtie^f J$. Then $(f(a)+j_1)(f(b)+j_2) \in K$ and so $(f(a)+j_1) \in \sqrt[T]{0_{R'}}$ or $(f(b)+j_2) \in K$. By Lemma 4.4, we have either $(a, f(a) + j_1) \in \overline{\sqrt[T]{0_{R'}}}^{f} \subseteq \sqrt[\bar{T}^f]{0_{R\bowtie^f J}}$ or $(b, f(b) + j_2) \in \bar{K}^f$ and the result follows. □

In particular, $S \times f(S)$ is a multiplicatively closed subset of $R\bowtie^f J$ for any multiplicatively closed subset S of R. Hence, we have the following corollary of Theorem 4.5.

Corollary 4.6 *Consider the amalgamation ring $R\bowtie^f J$ as above where f is an epimorphism. Let K be an ideal of R' and $T = f(S)$. Consider the following statements.*

1. $\bar{K}^f$ is a strongly $(S \times T)$-n-ideal of $R\bowtie^f J$.
2. $\bar{K}^f$ is a strongly $\bar{T}^f$-n-ideal of $R\bowtie^f J$.
3. K is a strongly T-n-ideal of R.

Then (1) ⇒ (2) ⇒ (3). Moreover, if $J \subseteq \sqrt{0_{R'}}$ and $Ker(f) \subseteq \sqrt{0_R}$, then the statements are equivalent.

We note that if $J \nsubseteq \sqrt{0_{R'}}$, then the equivalences in Theorems 4.2 and 4.5 are not true in general.

Example 4.7 Let $R = \mathbb{Z}$, $I = \langle 0\rangle = K$, $J = \langle 3\rangle \nsubseteq \sqrt{0_{\mathbb{Z}}} = 0_{\mathbb{Z}}$ and $S = \{1\} = T$. We have $I\bowtie J = \{(0, 3n) : n \in \mathbb{Z}\}$, $\bar{K} = \{(3n, 0) : n \in \mathbb{Z}\}$, $S\bowtie J = \{(1, 3n + 1) : n \in \mathbb{Z}\}$, $\bar{T} = \{(1 - 3n, 1) : n \in \mathbb{Z}\}$ and $\sqrt{0_{R\bowtie J}} = \{(0, 0)\}$.

1. I is a strongly S-n-ideal of R but $I\bowtie J$ is not a (strongly) $(S\bowtie J)$-n-ideal of $R\bowtie J$. Indeed, we have $(0, 3), (1, 4) \in R\bowtie J$ with $(0, 3)(1, 4) = (0, 12) \in I\bowtie J$. But clearly, $(0, 3) \notin \sqrt[S\bowtie J]{0_{R\bowtie J}}$ and $(1, 4) \notin I\bowtie J$.
2. K is a strongly T-n-ideal of R but $\bar{K}$ is not a (strongly) $\bar{T}$-n-ideal of $R\bowtie J$. For example, $(-3, 0), (-4, -1) \in R\bowtie J$ with $(-3, 0)(-4, -1) = (12, 0) \in \bar{K}$. However, $(-3, 0) \notin \sqrt[\bar{T}]{0_{R\bowtie J}}$ and $(-4, -1) \notin \bar{K}$.

Corollary 4.8 *Let I, K and J be ideals of a ring R and S be a multiplicatively closed subset of R.*

1. If $I\bowtie J$ is a strongly $(S\bowtie J)$-n-ideal of $R\bowtie J$, then I is a strongly S–n-ideal of R. Moreover, the converse is true if $J \subseteq \sqrt{0_R}$.

2. If $\bar{K}$ is a strongly $\bar{S}$-n-ideal of $R \bowtie J$, then K is a strongly T-n-ideal of R. The converse is true if $J \subseteq \sqrt{0_R}$.

Declarations

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References

1. Atiyah, M.F., MacDonald, I.G.: Introduction to Commutative Algebra, Addison-Wesley, Reading, MA. https://doi.org/10.1201/9780429493638 (1969)
2. Badawi, A.: On 2-absorbing ideals of commutative rings. Bull. Aust. Math. Soc. **75**(3), 417–429 (2017). https://doi.org/10.1017/S0004972700039344
3. Badawi, A., Tekir, U., Yetkin, E.: On weakly 2-absorbing primary ideals of commutative rings. J. Korean Math. Soc. **52**(1), 97–111 (2015). https://doi.org/10.4134/JKMS.2015.52.1.097
4. Badawi, A., Tekir, U., Ugurlu, E.A., Ulucak, G., Yetkin, Celikel E.: Generalizations of 2-absorbing primary ideals of commutative rings. Turk. J. Math. **40**(3), 703–717 (2016). https://doi.org/10.3906/mat-1505-43
5. Badawi, A., Tekir, U., Yetkin, E.: On 2-absorbing primary ideals in commutative rings. Bull. Korean Math. Soc. **51**(4), 1163–1173 (2014). https://doi.org/10.4134/BKMS.2014.51.4.1163
6. Călugăreanu, G.: UN-rings. J. Algeb. Appl. **15**(10), 1650182 (2016). https://doi.org/10.1142/S0219498816501826
7. D'Anna, M., Fontana, M.: An amalgamated duplication of a ring along an ideal: the basic properties. J. Algeb. Appl. **6**(3), 443–459 (2007). https://doi.org/10.1142/S0219498807002326
8. D'Anna, M., Fontana, M.: The amalgamated duplication of a ring along a multiplicative-canonical ideal. Arkiv für Matematik **45**(2), 241–252 (2007). https://doi.org/10.1007/s11512-006-0038-1
9. D'Anna, M., Finocchiaro, C.A., Fontana, M.: Properties of chains of prime ideals in an amalgamated algebra along an ideal. J. Pure Appl. Algeb. **214**(9), 1633–1641 (2010). https://doi.org/10.1016/j.jpaa.2009.12.008
10. Fuchs, L.: On quasi-primary ideals. Acta Sci. Math. **10**:11(3), 174–183 (1947)
11. Hamed, A., Malek, A.: S-prime ideals of a commutative ring. Beiträge Algeb. Geom. **61**(3), 533–542 (2020). https://doi.org/10.1007/s13366-019-00476-5
12. Khashan, H.A., Yetkin Çelikel, E.: S-n-ideals of commutative rings. Commun. Fac. Sci. Univ. Ankara Ser. A1 Math. Stat. **72**(1), 199–215 (2023). https://doi.org/10.31801/cfsuasmas.1099300
13. Khashan, H.A., Bani-Ata, B.B.: J-ideals of commutative rings. Int. Electron. J. Algeb. **29**(29), 148–164 (2020). https://doi.org/10.24330/ieja.852139
14. Koc, S., Uregen, R.N., Tekir, U.: On 2-absorbing quasi primary submodules. Filomat **31**(10), 2943–2950 (2017). https://doi.org/10.2298/FIL1710943K
15. Sharp, R.Y.: Steps in Commutative Algebra (No. 51). Cambridge university press, United Kingdom (2000). https://doi.org/10.1017/CBO9780511623684
16. Sevim, E.S., Tekir, U., Koc, S.: S-artinian rings and finitely s-cogenerated rings. J. Algeb. Appl. **19**(03), 2050051 (2020). https://doi.org/10.1142/S0219498820500516
17. Mohamadian, R.: r-ideals in commutative rings. Turk. J. Math. **39**(5), 733–749 (2015). https://doi.org/10.3906/mat-1503-35
18. Tamekkante, M., Bouba, E.M.: $(2, n)$-ideals of commutative rings. J. Algeb. Appl. **18**(06), 1950103 (2019). https://doi.org/10.1142/S0219498819501032
19. Tekir, U., Koc, S., Oral, K.H.: n-ideals of commutative rings. Filomat **31**(10), 2933–2941 (2017). https://doi.org/10.2298/FIL1710933T
20. Tekir, U., Koç, S., Oral, K.H., Shum, K.P.: On 2-absorbing quasi-primary ideals in commutative rings. Commun. Math. Stat. **4**(1), 55–62 (2016). https://doi.org/10.1007/S40304-015-0075-9
21. Yetkin Çelikel, E., Hamed, A.: Quasi-s-primary ideals of commutative rings. Comm. Algeb. **51**(10), 4285–4298 (2023). https://doi.org/10.1080/00927872.2023.2204970
22. Yıldız, E., Ersoy, B.A., Tekir, U., Koç, S.: On s-zariski topology. Comm. Algeb. **49**(3), 1212–1224 (2021). https://doi.org/10.1080/00927872.2020.1831006