fractal and fractional

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Third Hankel Determinant for the Logarithmic Coefficients of Starlike Functions Associated with Sine Function

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Abstract: The logarithmic functions have been used in a verity of areas of mathematics and other sciences. As far as we know, no one has used the coefficients of logarithmic functions to determine the bounds for the third Hankel determinant. In our present investigation, we first study some well-known classes of starlike functions and then determine the third Hankel determinant bound for the logarithmic coefficients of certain subclasses of starlike functions that also involve the sine functions. We also obtain a number of coefficient estimates. Some of our results are shown to be sharp.

Keywords: analytic functions; Hankel determinant; subordination; logarithmic coefficients; starlike functions

MSC: Primary 30C45; 30C50; 30C80

Citation: Khan, B.; Aldawish, I.; Araci, S.; Khan, M.G. Third Hankel Determinant for the Logarithmic Coefficients of Starlike Functions Associated with Sine Function. *Fractal Fract.* **2022**, *6*, 261. https://doi.org/10.3390/fractalfract6050261

Academic Editors: Acu Mugur Alexandru and Shahram Najafzadeh

Received: 22 March 2022
Accepted: 6 May 2022
Published: 9 May 2022

1. Introduction

We denote by $\mathcal{A}$ the class of analytic (holomorphic) functions f defined in the open unit disk

$$\mathbf{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

which satisfy the following normalization conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Thus, each $f \in \mathcal{A}$ has the following series form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathbf{U}. \tag{1}$$

Moreover, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ of functions which are univalent in $\mathbf{U}$. For two functions $h_1, h_2 \in \mathcal{A}$, we say that the function h_1 is subordinate to the function h_2 (written as $h_1 \prec h_2$) if there exists an analytic function w with the property

$$|w(z)| \leq |z| \quad \text{and} \quad w(0) = 0$$

such that

$$h_1(z) = h_2(w(z)) \qquad (z \in \mathbf{U}).$$

Moreover, if $h_2 \in \mathcal{S}$, then the above conditions can be written as:

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \text{ and } h_1(\mathbf{U}) \subset h_2(\mathbf{U}).$$

In 1992, Ma and Minda [1] introduced the class $\mathcal{S}^*(\Phi)$ as follows:

$$\mathcal{S}^*(\Phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \Phi(z) \right\}, \tag{2}$$

where the function Φ is assumed to be analytic with positive real part on $\mathbf{U}$ such that $\Phi(\mathbf{U})$ is axially symmetric and starlike with respect to

$$\Phi(0) = 1 \quad \text{and} \quad \Phi'(0) > 0.$$

Moreover, they investigated a number of useful geometric properties such as growth, distortion and covering results. By putting

$$\Phi(z) = (1+z)(1-z)^{-1}$$

specifically, then we can see that the functions class $\mathcal{S}^*(\Phi)$ is similar to that of the well-known class of starlike functions. For the various choices of the function Φ, we have the following function classes:

1. If we let
$$\Phi(z) = 1 + \sin z,$$
then we obtain the class
$$\mathcal{S}^*_{\sin} = \mathcal{S}^*(1 + \sin z),$$
of starlike functions whose image under an open unit disk is eight-shaped (see [2]).
2. For the choice
$$\Phi(z) = 1 + z - \frac{1}{3}z^3,$$
we obtain the class
$$\mathcal{S}^*_{nep} = \mathcal{S}^*\left(1 + z - \frac{1}{3}z^3\right),$$
whose image is bounded by a nephroid-shaped region (see [3]).
3. If we put
$$\Phi(z) = \sqrt{1+z},$$
then the function class leads to the class
$$\mathcal{S}^*_{\mathcal{L}} = \mathcal{S}^*\left(\sqrt{1+z}\right),$$
the class of starlike functions associated with the lemniscate of Bernoulli (see [4]).
4. Moreover, if we take
$$\Phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2,$$
we obtain the class
$$\mathcal{S}^*_{car} = \mathcal{S}^*\left(1 + \frac{4}{3}z + \frac{2}{3}z^2\right),$$
which is the class of starlike functions whose image under open unit is a cardioid shape and was introduced by Sharma et al. [5].
5. Furthermore, if we pick $\Phi(z) = e^z$ we obtain the class $\mathcal{S}^*_{\exp} = \mathcal{S}^*(e^z)$, which was introduced and studied by Mendiratta et al. [6].
6. If we put $\Phi(z) = \sqrt{1+z} + z$, then we have the class of starlike functions associated with the crescent-shaped region as discussed in [7].

The generalizations of the class $\mathcal{S}^*$ were studied by many authors. Indeed, they replaced Φ in (2) with Fibonacci numbers, Bell numbers, shell-like curves, conic domains and a modified sigmoid function [8–11], and they have defined some other generalized subclasses of the class of starlike functions.

It was Pommerenke [12,13] who studied the Hankel determinant $H_{q,n}(f)$ for a function $f \in \mathcal{A}$ written as in (1). The Hankel determinant $H_{q,n}(f)$ is given as follows:

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{3}$$

For different values of q and n, the Hankel determinants for various orders are derived. For example, when $n = 1$ and $q = 2$, the above-defined determinant becomes as follows:

$$|H_{2,1}(f)| \quad = \quad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \left|a_3 - a_2^2\right|, \quad \text{where } a_1 = 1.$$

We note that the nth coefficient of a function class $\mathcal{S}$ is well known to be bounded by n, and the coefficient limits give information about the function's geometric characteristics. The famous problem solved by Fekete–Szegö [14] is to determine the greatest value of the coefficient functional $|a_3 - \sigma a_2^2|$ over the class $\mathcal{S}$ for each $\sigma \in [0,1]$, which was demonstrated using the Loewner technique. For a detailed study about this well-known functional, see [15–17]. Furthermore, if we take $q = n = 2$, then we have the second Hankel determinant

$$H_{2,2}(f) \quad = \quad \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

In recent years, many authors have studied and investigated the upper bound of $|H_{2,2}(f)|$ for different subclasses of analytic functions. A few of them are Noonan and Thomas [18], Hayman [19], Ohran et al. [20] and Shi et al. [21]. Furthermore, the bounds for the third Hankel determinant were first investigated by Babalola [22]. Some recent and interesting works on this topic maybe found in [23–26].

In [2], Cho et al. defined and studied a class of starlike functions associated with the sine function, defined as follows:

$$\mathcal{S}^*_{\sin} = \left\{ f \in \mathcal{S} : \frac{zf'(z)}{f(z)} \prec 1 + \sin(z) \right\} \quad (z \in D). \tag{4}$$

The logarithmic coefficients of $f \in \mathcal{S}$, denoted by $\gamma_n = \gamma_n(f)$, are defined by the following series expansion:

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n z^n.$$

Logarithmic coefficients have recently attracted considerable interest. For instance, Milin's conjecture highly depends on logarithmic coefficients (see [27]; see also ([28], page 155)). Ali et al. [29] investigated the logarithmic coefficients of some close-to-convex functions, while the third logarithmic coefficient in some subclasses of close-to-convex functions was studied by Cho et al. [30]. Moreover, logarithmic coefficients of univalent functions can be found in [31]. Very recently, Kowalczyk and Lecko [32] have studied the Hankel matrices whose entries are logarithmic coefficients of univalent functions and have given sharp bounds for the second Hankel determinant of logarithmic coefficients of convex and starlike functions. For some other related works, see [33–35]. For a function f given by (1), the logarithmic coefficients are as follows:

$$\gamma_1 = \frac{1}{2}a_2, \tag{5}$$

$$\gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{6}$$

$$\gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right), \tag{7}$$

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right), \tag{8}$$

$$\gamma_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right). \tag{9}$$

Based on all of the above ideas, we propose the study of the Hankel determinant, whose entries are logarithmic coefficients of $f \in \mathcal{S}$, that is

$$H_{q,n}(f) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2q-2} \end{vmatrix}. \tag{10}$$

The main aim of this paper is to find upper bounds for $H_{3,1}(f)$ for the class of starlike functions associated with the sine functions.

2. A Set of Lemmas

We denote by $\mathcal{P}$ the class of analytic functions p which are normalized by

$$p(0) = 1 \quad \text{with} \quad \Re(p(z)) > 0 \quad (z \in \mathbf{U})$$

and have the following form:

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (z \in \mathbf{U}). \tag{11}$$

To prove our main results, we need the following lemmas.

Lemma 1. *([36]) Let $p \in \mathcal{P}$. Then, there exist x, δ with $|x| \le 1, |\delta| \le 1$ such that*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{12}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\delta. \tag{13}$$

Lemma 2. *If $p \in \mathcal{P}$, then the following inequalities hold*

$$|c_k| \le 2 \text{ for } k \ge 1, \tag{14}$$

$$|c_{n+k} - \mu c_n c_k| < 2 \text{ for } 0 \le \mu \le 1, \tag{15}$$

$$|c_m c_k - c_k c_1| \le 4 \text{ for } m + k = k + l, \tag{16}$$

$$\left|c_{k+2k} - \mu c_k c_k^2\right| \le 2(1 + 2\mu), \text{ for } \mu \in \mathbb{R}, \tag{17}$$

$$\left|c_2 - \frac{c_1^2}{2}\right| \le 2 - \frac{|c_1^2|}{2}, \tag{18}$$

and for complex number η, we have

$$\left|c_2 - \eta c_1^2\right| < 2\max\{1, |2\eta - 1|\}. \tag{19}$$

For the inequalities (14)–(18), see [13], and (19) is given in [15].

Lemma 3. *([37], Lemma 2.2) If $p \in \mathcal{P}$, then*

$$|Ic_1^3 - Xc_1c_2 + Vc_3| \leq 2|I| + 2|X - 2I| + 2|I - X + V|, \tag{20}$$

where I, X and V are real numbers.

3. Main Results

Theorem 1. *If $f \in \mathcal{S}^*_{\sin}$ and it has the form given in (1), then*

$$\begin{aligned}
|\gamma_1| &\leq \frac{1}{2}, && (21)\\
|\gamma_2| &\leq \frac{1}{4}, && (22)\\
|\gamma_3| &\leq \frac{1}{6}, && (23)\\
|\gamma_4| &\leq \frac{1}{8}, && (24)\\
|\gamma_5| &\leq \frac{7}{10}. && (25)
\end{aligned}$$

The following functions are examples for the sharpness of the above first four inequalities

$$\begin{aligned}
f_1(z) &= z\exp\left(\int_0^z \frac{\sin(t)}{t}dt\right) = z + z^2 + \cdots, && (26)\\
f_2(z) &= z\exp\left(\int_0^z \frac{\sin(t^2)}{t}dt\right) = z + \frac{1}{2}z^3 + \cdots, && (27)\\
f_3(z) &= z\exp\left(\int_0^z \frac{\sin(t^3)}{t}dt\right) = z + \frac{1}{3}z^4 + \cdots. && (28)\\
f_4(z) &= z\exp\left(\int_0^z \frac{\sin(t^4)}{t}dt\right) = z + \frac{1}{4}z^5 + \cdots. && (29)
\end{aligned}$$

respectively.

Proof. Let $f \in \mathcal{S}^*_{\sin}$ and then, by the definitions of subordinations, there exists a Schwartz function $w(z)$ with the properties that

$$w(0) = 0 \quad \text{and} \quad w(z) < 1,$$

such that

$$\frac{zf'(z)}{f(z)} = 1 + \sin(w(z)) \tag{30}$$

Define the function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots.$$

It is clear that $p(z) \in \mathcal{P}$. This implies that

$$w(z) = \frac{p(z)-1}{p(z)+1} = \frac{c_1 z + c_2 z^2 + c_3 z^3 + \cdots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}$$
$$= \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{1}{8}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right)z^3 + \cdots.$$

Now, from (30), we have

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 2a_2 a_3 + 3a_4\right)z^3 + \left(4a_5 - a_2^4 + 4a_2^2 a_3 - 4a_2 a_4 - 2a_3^2\right)z^4 + \cdots \tag{31}$$

and

$$1 + \sin(w(z)) = 1 + \frac{1}{2}c_1 z + \left(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\right)z^2 + \left(\frac{5}{48}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3\right)z^3 + \left(\frac{1}{2}c_4 - \frac{1}{2}c_1 c_3 + \frac{5}{16}c_1^2 c_2 - \frac{1}{4}c_2^2 - \frac{c_1^4}{32}\right)z^4 + \cdots. \tag{32}$$

Comparing (31) and (32), we achieve

$$a_2 = \frac{c_1}{2}, \tag{33}$$
$$a_3 = \frac{c_2}{4}, \tag{34}$$
$$a_4 = \frac{c_3}{6} - \frac{c_1 c_2}{24} - \frac{c_1^3}{144}, \tag{35}$$
$$a_5 = \frac{c_4}{8} - \frac{c_1 c_3}{24} + \frac{5c_1^4}{1152} - \frac{c_1^2 c_2}{192} - \frac{c_2^2}{32}, \tag{36}$$
$$a_6 = \frac{-3}{80}c_1 c_4 - \frac{7}{120}c_2 c_3 - \frac{11}{4800}c_1^5 - \frac{43}{960}c_1 c_2^2 + \frac{71}{5760}c_1^3 c_2 + \frac{c_5}{10}. \tag{37}$$

Now, from (5) to (9) and (33) to (37), we obtain

$$\gamma_1 = \frac{1}{4}c_1, \tag{38}$$
$$\gamma_2 = \frac{1}{8}c_2 - \frac{1}{16}c_1^2, \tag{39}$$
$$\gamma_3 = \frac{5}{288}c_1^3 - \frac{1}{12}c_1 c_2 + \frac{1}{12}c_3, \tag{40}$$
$$\gamma_4 = \frac{1}{16}c_4 - \frac{1}{16}c_1 c_3 + \frac{9}{128}c_1^2 c_2 - \frac{1}{32}c_2^2 - \frac{17}{2304}c_1^4, \tag{41}$$
$$\gamma_5 = \frac{1}{38\,400}c_1^5 - \frac{1}{80}c_1^3 c_2 + \frac{1}{32}c_3 c_1^2 + \frac{1}{160}c_1 c_2^2 - \frac{1}{20}c_4 c_1 - \frac{1}{20}c_3 c_2 + \frac{1}{20}c_5. \tag{42}$$

Applying (14) to (38), we get

$$|\gamma_1| \leq \frac{1}{2}.$$

From (39) and using (18), we have

$$|\gamma_2| = \frac{1}{8}\left|c_2 - \frac{1}{2}c_1^2\right| \leq \frac{1}{8}\left(2 - \frac{|c_1|^2}{2}\right) = H(c_1).$$

Clearly, $H(c_1)$ is a decreasing function and its maximum is attained at $c_1 = 0$, hence

$$|\gamma_2| \leq \frac{1}{4}.$$

Applying Lemma 3 on Equation (40), we get

$$|\gamma_3| \leq \frac{1}{6}.$$

Moreover, using Lemma 3 on (41), we get

$$|\gamma_4| \leq \frac{1}{8}.$$

Rearranging (42), we obtain

$$\begin{aligned}\gamma_5 &= \frac{-1}{80}c_1^3\left(c_2 - \frac{1}{480}c_1^2\right) - \frac{1}{20}c_1\left(c_4 - \frac{5}{8}c_1c_3\right) \\ &\quad -\frac{1}{20}c_2\left(c_3 - \frac{1}{8}c_1c_2\right) + \frac{1}{20}c_5.\end{aligned}$$

By making use of (14) and (15), along with the triangular inequality, we can easily obtain the desired result.

To prove the sharpness of (21) to (24), observe that

$$\begin{aligned}\log\frac{f_1(z)}{z} &= 2\sum_{n=2}^{\infty}\gamma(f_1)z^n = z - \frac{1}{18}z^3 + \cdots, \\ \log\frac{f_2(z)}{z} &= 2\sum_{n=2}^{\infty}\gamma(f_2)z^n = \frac{1}{2}z^2 + \cdots, \\ \log\frac{f_3(z)}{z} &= 2\sum_{n=2}^{\infty}\gamma(f_3)z^n = \frac{1}{3}z^3 + \cdots, \\ \log\frac{f_4(z)}{z} &= 2\sum_{n=2}^{\infty}\gamma(f_4)z^n = \frac{1}{4}z^4 + \cdots.\end{aligned}$$

It follows that these inequalities are sharp. □

Theorem 2. *If $f \in \mathcal{S}^*_{\sin}$ and it has the form given in (1), then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{16}. \tag{43}$$

The function f_2 given in (27) is an example of sharpness for this result.

Proof. From (38)–(40), we obtain

$$\gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{2304}c_1^4 - \frac{1}{192}c_1^2c_2 + \frac{1}{48}c_3c_1 - \frac{1}{64}c_2^2.$$

Using Lemma 1 to write c_2 and c_3 in terms of $c_1 = c \in [0,2]$, we have

$$\begin{aligned}\gamma_1\gamma_3 - \gamma_2^2 &= -\frac{1}{1152}c^4 - \frac{1}{256}(4-c^2)^2x^2 \\ &\quad -\frac{1}{192}c^2(4-c^2)x^2 + \frac{1}{96}c(4-c^2)(1-|x|^2)\delta.\end{aligned}$$

Applying triangle inequality and using $|\delta| \leq 1$ and $|x| = y \leq 1$, we get

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{1152}c^4 + \frac{1}{256}(4-c^2)^2y^2 + \frac{1}{192}c^2(4-c^2)y^2 + \frac{1}{96}c(4-c^2)(1-y^2) = G(c,y) \quad \text{say.}$$

Now, differentiating partially with respect to y, we achieve

$$\frac{\partial G(c,y)}{\partial y} = \frac{1}{128}(4-c^2)^2y + \frac{1}{96}c^2(4-c^2)y - \frac{1}{48}c(4-c^2)y.$$

Clearly, $\frac{\partial G(c,y)}{\partial y} > 0$ and then $G(c,y)$ is increasing in y for fixed c. For this reason, $G(c,y)$ attains its maximum at $y = 1$, so

$$\begin{aligned} G(c,y) &\leq G(c,1) = \frac{1}{1152}c^4 + \frac{1}{256}(4-c^2)^2 + \frac{1}{192}c^2(4-c^2) \\ &= -\frac{1}{2304}c^4 - \frac{1}{96}c^2 + \frac{1}{16}. \end{aligned}$$

Now, differentiating with respect to c, we have

$$G^{'}(c,1) = -\frac{1}{576}c^3 - \frac{1}{48}c.$$

Clearly, $G'(c,1) \leq 0$, is a decreasing function so, at $c = 0$, the maximum value is attained, that is

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{16}$$

□

Theorem 3. *If $f \in \mathcal{S}^*_{\sin}$ and it has the form given in (1), then*

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \leq \frac{53}{288}. \tag{44}$$

Proof. From (38)–(40), we get

$$\gamma_2\gamma_4 - \gamma_3^2 = \frac{53}{331\,776}c_1^6 - \frac{67}{27\,648}c_1^4c_2 + \frac{7}{6912}c_1^3c_3 + \frac{35}{9216}c_1^2c_2^2 - \frac{1}{256}c_4c_1^2 + \frac{7}{1152}c_1c_2c_3 - \frac{1}{256}c_2^3 + \frac{1}{128}c_4c_2 - \frac{1}{144}c_3^2.$$

Rearranging the above, we have

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| = \left|-\frac{67}{27648}c_1^4\left(c_2 - \frac{53}{804}c_1^2\right) - \frac{1}{256}c_1^2\left(c_4 - \frac{35}{36}c_2^2\right) + \frac{1}{128}c_2\left(c_4 - \frac{1}{2}c_2^2\right) + \frac{7}{6912}c_1^3c_3 - \frac{1}{144}c_3\left(c_3 - \frac{7}{8}c_1c_2\right)\right|.$$

Applying triangle inequality, we get

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \leq \frac{67}{27648}|c_1|^4\left|c_2 - \frac{53}{804}c_1^2\right| + \frac{1}{256}|c_1|^2\left|c_4 - \frac{35}{36}c_2^2\right| + \frac{1}{128}|c_2|\left|c_4 - \frac{1}{2}c_2^2\right| + \frac{7}{6912}|c_1|^3|c_3| + \frac{1}{144}|c_3|\left|c_3 - \frac{7}{8}c_1c_2\right|.$$

Using (14) and (15), we get the required result. □

Theorem 4. *If* $f \in \mathcal{S}^*_{\sin}$ *and it has the form given in (1), then*

$$|\gamma_1\gamma_4 - \gamma_2\gamma_3| \le \frac{77}{288}. \tag{45}$$

Proof. From (38)–(40), we get

$$\gamma_1\gamma_4 - \gamma_2\gamma_3 = -\frac{7}{9216}c_1^5 + \frac{47}{4608}c_1^3c_2 - \frac{1}{96}c_3c_1^2 + \frac{1}{64}c_4c_1 + \frac{1}{384}c_1c_2^2 - \frac{1}{96}c_3c_2.$$

Rearranging, we get

$$|\gamma_1\gamma_4 - \gamma_2\gamma_3| = \left|\frac{47}{4608}c_1^3\left(c_2 - \frac{7}{94}c_1^2\right) + \frac{1}{64}c_1\left(c_4 - \frac{2}{3}c_1c_3\right) - \frac{1}{96}c_2\left(c_3 - \frac{1}{4}c_1c_2\right)\right|.$$

Applying triangle inequality, we get

$$|\gamma_1\gamma_4 - \gamma_2\gamma_3| \le \frac{47}{4608}|c_1|^3\left|c_2 - \frac{7}{94}c_1^2\right| + \frac{1}{64}|c_1|^2 \cdot \left|c_4 - \frac{2}{3}c_1c_3\right| + \frac{1}{96}|c_2|\left|c_3 - \frac{1}{4}c_1c_2\right|.$$

Using (14) and (15), we get the required result. □

Theorem 5. *If* $f \in \mathcal{S}^*_{\sin}$ *and it has the form given in (1), then*

$$|H_{3,1}(f)| \le \frac{3727}{34\,560} \simeq 0.107\,84$$

Proof. Since

$$\begin{aligned} |H_{3,1}(f)| &= \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{vmatrix} \\ &\le |\gamma_3|\left|\gamma_2\gamma_4 - \gamma_3^2\right| + |\gamma_4||\gamma_1\gamma_4 - \gamma_2\gamma_3| + |\gamma_5|\left|\gamma_1\gamma_3 - \gamma_2^2\right|. \end{aligned}$$

From the values of (23)–(25), (43)–(45), we achieve the required result. □

4. Concluding Remarks and Observations

Here, in our present investigation, we have successfully examined and studied some well-known subclasses of starlike functions associated with various domains. We have then obtained a number of coefficient estimates and the third-order Hankel determinant bound for the logarithmic coefficients of starlike functions that are associated with the sine functions. We have also given some examples to show that some of our results are sharp.

The study of coefficient problems (such as the Fekete–Szegö and the Hankel determinant problems) continues to inspire scholars in the Geometric Function Theory of Complex Analysis. We have chosen to include many recent works (see, for example, [38–44]), on various bi-univalent function classes, as well as ongoing uses of the q-calculus in the study of other analytic or meromorphic univalent and multivalent function classes in order to provide incentive and motivation to interested readers.

Author Contributions: Conceptualization, B.K., I.A., S.A. and M.G.K.; writing—original draft preparation, B.K., I.A., S.A. and M.G.K; writing—review and editing, B.K., I.A., S.A. and M.G.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: No data were used to support this study.

Acknowledgments: The authors are thankful to the reviewers for their many valuable suggestions and recommendations.

Conflicts of Interest: The authors declare that they have no competing interests.

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