



THE TRIPLE ZERO GRAPH OF A COMMUTATIVE RING

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ABSTRACT. Let R be a commutative ring with non-zero identity. We define the set of triple zero elements of R by $TZ(R) = \{a \in Z(R)^* : \text{there exist } b, c \in R \setminus \{0\} \text{ such that } abc = 0, ab \neq 0, ac \neq 0, bc \neq 0\}$. In this paper, we introduce and study some properties of the triple zero graph of R which is an undirected graph $TZ\Gamma(R)$ with vertices $TZ(R)$, and two vertices a and b are adjacent if and only if $ab \neq 0$ and there exists a non-zero element c of R such that $ac \neq 0, bc \neq 0$, and $abc = 0$. We investigate some properties of the triple zero graph of a general ZPI-ring R , we prove that $\text{diam}(TZ\Gamma(R)) \in \{0, 1, 2\}$ and $\text{gr}(TZ\Gamma(R)) \in \{3, \infty\}$.

1. INTRODUCTION

Throughout this paper, all rings are commutative with identity and $Z(R)$ denotes the set of zero-divisors of a ring R . The concept of the zero-divisor graph of a commutative ring was introduced by I. Beck [9]. He let all elements of R be vertices of the graph and his work was mostly concerned with coloring of rings. In [3], all elements of a commutative ring R are vertices, and distinct vertices a and b are adjacent if and only if $ab = 0$. This graph is denoted by $\Gamma_0(R)$. Then D.F. Anderson and P.S. Livingston [4] introduced a (induced) zero-divisor subgraph $\Gamma(R)$ of $\Gamma_0(R)$. The zero-divisor graph $\Gamma(R)$ introduced in [13] and [4] is as follows: Two distinct vertices $x, y \in Z(R)^* = Z(R) \setminus \{0\}$ are adjacent if and only if $xy = 0$. In [4], D.F. Anderson and P.S. Livingston have shown that $\Gamma(R)$ is connected with $\text{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\text{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. The zero-divisor graph of a commutative ring in the sense of Anderson–Livingston has been studied extensively by several authors, [1], [2], [5], [6], [14], [15]. Since then, the concept of the zero-divisor graph of ring has been playing a vital role in its expansion.

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We define the set of the triple zero elements of R by $TZ(R) = \{a \in Z(R)^* : \text{there exist } b, c \in R \setminus \{0\} \text{ such that } abc = 0, ab \neq 0, ac \neq 0, bc \neq 0\}$. It is clear that every triple zero element of R is a zero-divisor of R , but the converse is not true in general. For example, the element 2 is a zero-divisor of \mathbb{Z}_6 , but clearly it is not a triple zero element. In this paper, motivated from zero-divisor graphs, we introduce the triple zero graph of a commutative ring. Our starting point is the following definition: The triple zero graph of R is an undirected graph $TZ\Gamma(R)$ with vertices $TZ(R)$. If two distinct elements a and b are adjacent, then (a, b) is an edge and we will denote it by $a \sim b$. Two distinct vertices a and b are adjacent if and only if $ab \neq 0$ and there exists an element $c \in R \setminus \{0\}$ such that $ac \neq 0, bc \neq 0$ and $abc = 0$. The relation " \sim " is always symmetric, but neither reflexive nor transitive in general. For instance, let $R = \mathbb{Z}_{36}$. Then clearly $2, 3, 6 \in TZ(R)$ with $6 \approx 6$, and also $2 \sim 3, 2 \sim 9$, but $3 \not\sim 9$.

Recall from [8] that I is said to be a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. As defined in [7], I is said to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I, ac \in I$, or $bc \in I$. From these definitions, note that $\{0\}$ is always a weakly 2-absorbing ideal of R . If 0 is not a 2-absorbing ideal, then there are some triple zero elements of R . The concept of (weakly) 2-absorbing ideals and the zero-divisor graphs motivated us to define the triple zero divisor graph and also investigate the relations between triple zero graph of a ring R and 2-absorbing ideals of R .

Among many results in this paper, in Section 2, we justify some properties of the triple zero graph of commutative rings. In Theorem 1, we show that a proper ideal I of a ring R is 2-absorbing if and only if $TZ\Gamma(R/I) = \emptyset$. In Theorem 11, we characterize triangle free triple zero graphs of general ZPI-rings. In [11], the authors define 3-zero-divisor hypergraph regarding to an ideal with vertices $\{x \in R \setminus I : xyz \in I \text{ for some } y, z \in R \setminus I \text{ such that } xy \notin I, yz \notin I, xz \notin I\}$ where distinct vertices are adjacent if and only if $xyz \in I, xy \notin I, yz \notin I$ and $xz \notin I$. They conclude that diameter of this graph is at most 4. In Section 3, we study the triple zero graph of general ZPI-rings. The graph properties of the triple zero graph of general ZPI-rings such as diameter and girth are investigated. We obtain that the triple zero graph of a zero dimensional general ZPI-ring is always connected with diameter at most 2 and girth 3 if it is determined. (Corollary 12). Furthermore, we give some characterizations for the triple zero graph of \mathbb{Z}_n where $n > 1$ and justify the diameter and girth of $TZ\Gamma(\mathbb{Z}_n)$. (Theorem 13, Theorem 14 and Corollary 15)

For the sake of completeness, we state some definitions and notation used throughout. Let G be a (undirected) graph. The order of G , denoted by $|G|$, is equal to the cardinality of the vertex set. The graph G is connected if there is a path between any two distinct vertices. For vertices a and b of G , we say that the distance between a and b , $d(a, b)$ is the length of a shortest path from a to b . If there is no path between a and b , then $d(a, b) = \infty$, and $d(a, a) = 0$. A graph G is said to be totally disconnected if it has no edges. The diameter of G is defined by $diam(G) = \sup\{d(a, b) : a$

and b are vertices of G }. The girth of G , denoted by $gr(G)$, is the length of a shortest cycle in G . If G contains no cycles, then $gr(G) = \infty$. A cycle of length three is commonly called a triangle. A triangle-free graph is an undirected graph in which no three vertices form a triangle of edges. A graph G is complete if any two distinct vertices are adjacent. The complete graph with n vertices will be denoted by K_n . A complete bipartite graph is a graph G which may be partitioned into two disjoint non-empty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. We denote the complete bipartite graph by $K_{m,n}$ where A and B are partitions with $|A| = m$ and $|B| = n$. If one of the vertex sets is a singleton, then we call G a star graph. A star graph is clearly $K_{1,n}$. As usual, \mathbb{Z} and \mathbb{Z}_n will denote the integers and integers modulo n , respectively. For general background and terminology, the reader may consult [10].

2. PROPERTIES OF THE TRIPLE ZERO GRAPH

Theorem 1. *Let R be a commutative ring and I be a proper ideal of R . Then the following statements hold:*

- (1) $TZ\Gamma(R/I) = \emptyset$ if and only if I is a 2-absorbing ideal of R .
- (2) $TZ\Gamma(R) = \emptyset$ if and only if $\{0\}$ is a 2-absorbing ideal of R .
- (3) If (R, M) is a quasi-local ring with $M^2 = 0$, then $TZ\Gamma(R) = \emptyset$.

Proof. Suppose that I is not a 2-absorbing ideal of R . Then there exist some (not necessarily distinct) elements a, b, c of R with $abc \in I$ but neither $ab \in I$ nor $ac \in I$ nor $bc \in I$. Hence $(a+I)(b+I)(c+I) = I$ but neither $(a+I)(b+I) = I$ nor $(a+I)(c+I) = I$ nor $(b+I)(c+I) = I$. Thus $a, b, c \in TZ(R/I)$; and so $TZ\Gamma(R/I) \neq \emptyset$. Conversely, if $TZ\Gamma(R/I) \neq \emptyset$, then there are some (not necessarily distinct) elements $a+I, b+I, c+I$ of R/I satisfying $(a+I)(b+I)(c+I) = I$ but neither $(a+I)(b+I) = I$ nor $(a+I)(c+I) = I$ nor $(b+I)(c+I) = I$. It implies that $ab, ac, bc \notin I$ and $abc \in I$. Hence I is not a 2-absorbing ideal of R .

(2) It is clearly a particular case putting $I = 0$ in (1).

(3) Suppose that (R, M) is a quasi-local ring with $M^2 = 0$. Hence 0 is a 2-absorbing ideal of R by [7, Corollary 3.3]. Thus $TZ\Gamma(R) = \emptyset$ by (2). \square

The following example shows that the converse of Theorem 1 (3) does not hold.

Example 2. *Consider $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then clearly $TZ\Gamma(R) = \emptyset$ but since R has two maximal ideals $0 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times 0$, it is not a quasi-local ring.*

Let $R = \mathbb{Z}_p[X]/\langle X^n \rangle$, where p is prime and $n \geq 3$. We denote $a(X)$ as the congruence class of polynomials congruent to $a(X) \pmod{\langle X^n \rangle}$. It is well-known that an element of $\mathbb{Z}_p[X]/\langle X^n \rangle$ is of the form $a(X) = a_0 + a_1X + a_2X^2 + \cdots + a_kX^k$ of degree $k \leq n$ where $a_i \in \mathbb{Z}_p$ for $i \in \{1, 2, \dots, k\}$. Now we determine the vertex set of the graph $TZ(\mathbb{Z}_p[X]/\langle X^n \rangle)$.

Theorem 3. Let $a(X) = a_0 + a_1X + a_2X^2 + \dots + a_kX^k \in \mathbb{Z}_p[X]/\langle X^n \rangle$ where $n \geq 3$. Then $a(X)$ is a vertex of the graph $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^n \rangle)$ if and only if $a_0 = 0 \pmod{p}$ and of the form in one of the following types:

- (1) $a_1 = a_2 = \dots = a_{k-1} = 0$ and $k \leq n - 2$.
- (2) $a_i \neq 0$ for some $r = 1, 2, \dots, k - 1$ and $k \leq n - 1$.

Proof. Let $a(X) \in TZ(\mathbb{Z}_p[X]/\langle X^n \rangle)$. Then there exists non-zero $b(X), c(X) \in \mathbb{Z}_p[X]/\langle X^n \rangle$ such that $a(X)b(X)c(X) = 0 \pmod{\langle X^n \rangle}$, $a(X)b(X) \neq 0 \pmod{\langle X^n \rangle}$, $a(X)c(X) \neq 0 \pmod{\langle X^n \rangle}$ and $b(X)c(X) \neq 0 \pmod{\langle X^n \rangle}$. Let $b(X) = b_0 + b_1X + b_2X^2 + \dots + b_tX^t$, $c(X) = c_0 + c_1X + c_2X^2 + \dots + c_sX^s$ where b_j and c_r are the first non-zero (i.e., $b_j, c_r \neq 0 \pmod{p}$) coefficients in the polynomials $b(X)$ and $c(X)$, respectively. Then the coefficient of X_{j+r} in the product $a(X)b(X)c(X)$ is $a_0b_jc_r$. Since $a(X)b(X)c(X) = 0 \pmod{\langle X^n \rangle}$ and $j, r < n$, we must have $a_0b_jc_r = 0 \pmod{p}$. Observe that since b_j, c_r are non-zero elements of \mathbb{Z}_p , we have $b_jc_r \neq 0$. Thus $a_0 = 0 \pmod{p}$.

Case I. Suppose that $a_1 = a_2 = \dots = a_{k-1} = 0$. Then $a_kX^kb_jX^jc_rX^r = 0 \pmod{\langle X^n \rangle}$ which implies that $k + j + r = n$. Since $j, r \geq 1$, we conclude that $k \leq n - 2$.

Case II. Suppose that $a_i \neq 0$ for some $i = 1, 2, \dots, k - 1$. Then we show that k can be $n - 1$. Assume that $\deg(a(X)) = k = n - 1$. Then, clearly $a(X)X = 0 \pmod{\langle X^n \rangle}$ and $X^2 \neq 0 \pmod{\langle X^n \rangle}$. Since $a_iX^iX \neq 0 \pmod{\langle X^n \rangle}$ where $i = 1, 2, \dots, k - 1$, we conclude that $a(X)X \neq 0 \pmod{\langle X^n \rangle}$.

Conversely, assume that $a_0 = 0 \pmod{p}$. If (1) holds, then $a(X) = a_kX^k$ and $k \leq n - 2$. Then $a(X)X^jX^r = 0 \pmod{\langle X^n \rangle}$ for all $j, r \geq 1$ such that $j + r = n - k$ but neither $a(X)X^j = 0 \pmod{\langle X^n \rangle}$ nor $a(X)X^r = 0 \pmod{\langle X^n \rangle}$ nor $X^jX^r = 0 \pmod{\langle X^n \rangle}$. Hence $a(X)$ is a triple zero element of $\mathbb{Z}_p[X]/\langle X^n \rangle$. Suppose that (2) holds. We may assume that $a_1 \neq 0 \pmod{p}$. Then $a(X)X^jX^r = 0 \pmod{\langle X^n \rangle}$ for all $j, r \geq 1$ such that $j + r = n - 1$. Since $a_1X^jX^r \neq 0 \pmod{\langle X^n \rangle}$ and $a_1X^jX^r \neq 0 \pmod{\langle X^n \rangle}$, we conclude that $a(X)X^j \neq 0 \pmod{\langle X^n \rangle}$ and $a(X)X^r \neq 0 \pmod{\langle X^n \rangle}$. Thus $a(X)$ is a triple zero element of $\mathbb{Z}_p[X]/\langle X^n \rangle$. \square

Theorem 4. Let $R = \mathbb{Z}_p[X]/\langle X^3 \rangle$. Then $TZ\Gamma(R)$ is a complete graph with $p^2 - p$ vertices, i.e., $TZ\Gamma(R) \cong K_{p^2-p}$. In particular, if $p = 2$, then $TZ\Gamma(R) \cong K_2$.

Proof. From Theorem 3, the vertices of $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^3 \rangle)$ of the type $nX + mX^2$, where n, m are integers with $1 \leq n \leq p$ and $0 \leq m \leq p$. Hence, the number of the vertices of $TZ\Gamma(\mathbb{Z}_p[X]/\langle X^3 \rangle)$ is $p^2 - p$. Observe that all vertices of this graph are adjacent, thus it is the complete graph K_{p^2-p} . For $p = 3$, this graph is illustrated by Figure 2. In the particular case, since $X^2X(X + X^2) = 0$ but $X^2X \neq 0$ and $X(X + X^2) \neq 0$, X and $(X + X^2)$ are the only distinct adjacent vertices of $TZ\Gamma(\mathbb{Z}_2[X]/\langle X^3 \rangle)$. \square

We are unable to answer the following question which may be inspiring for the possible other work:

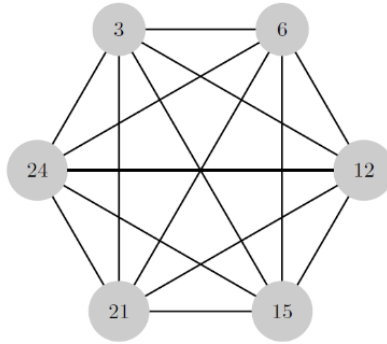


FIGURE 1. $TZ\Gamma(\mathbb{Z}_{27})$

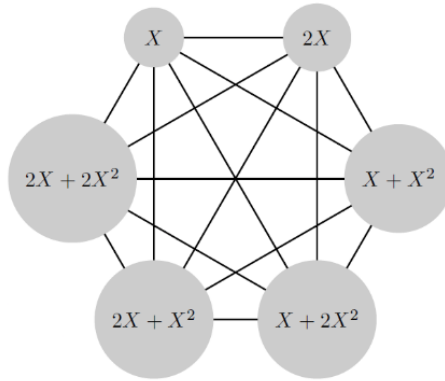


FIGURE 2. $TZ\Gamma(\mathbb{Z}_3[X]/\langle X^3 \rangle)$

Question. Let $R = \mathbb{Z}_p[X]/\langle X^n \rangle$ where p is a prime number and $n \geq 3$. Can we have a general characterization for the triple zero graph of R ?

We recall that an n -gon is a regular polygon with n sides. In the next example, we show that there are triple zero graphs with cycles of arbitrary specified length.

Example 5. Let T be an integral domain and $n \geq 3$ is an integer. Consider $R = T[X_1, X_2, \dots, X_n]/(X_1X_2X_3, X_3X_4X_5, \dots, X_{n-1}X_nX_1)$. Then $TZ\Gamma(R)$ is a connected graph which has an n -gon, an $n/2$ -gon and has triangles more than n .

Proof. Observe that $X_1 \sim X_2 \sim X_3$, $X_3 \sim X_4 \sim X_5, \dots, X_{n-1} \sim X_n \sim X_1$ are some of the triangles, and it is easy to see that $(X_k + X_kX_{k+1}) \sim (X_{k+1} + X_kX_{k+1}) \sim X_{k+2}$ is another triangle for each k , where k is odd, and $k < n - 2$. Also $X_1 \sim X_3 \sim \dots \sim X_{n-1} \sim X_1$ is an $n/2$ -gon and $X_1 \sim X_2 \sim \dots \sim X_{n-1} \sim X_n \sim X_1$ is an n -gon. \square

3. TRIPLE ZERO GRAPH OF GENERAL ZPI-RINGS

A ring is called a general ZPI-ring (resp. ZPI-ring) if each ideal (resp. each non-zero ideal) I of R is uniquely expressible as product of prime ideals of R . Dedekind domains are indecomposable general ZPI-rings. For a general background, the reader may refer to [12]. In this section, we study the graph theoretical properties of the triple zero graph for general ZPI-rings. First we need to prove the following lemma which is a generalization of [8, Theorem 3.15].

Lemma 6. Let R be a zero dimensional Noetherian ring which is not a field. Then the following statements are equivalent:

- (1) R is a general ZPI-ring.
- (2) If I is a 2-absorbing ideal of R , then I is a maximal ideal of R or $I = M^2$ for some maximal ideal M of R or $I = MM'$ for some maximal ideals M, M' of R .
- (3) If I is a 2-absorbing ideal of R , then I is a prime ideal of R or $I = P^2$ for some prime ideal P of R or $I = PQ$ for some prime ideals P, Q of R .

Proof. (1) \Rightarrow (2) Let I be a 2-absorbing ideal of R . Since maximal ideals coincide with prime ideals, $\sqrt{I} = M$ for some maximal ideal M of R with $M^2 \subseteq I$ or $\sqrt{I} = M \cap M' = MM'$ for some maximal ideals M, M' of R with $MM' \subseteq I$ by [8, Theorem 2.4]. Thus, we have either $I = M$ is maximal or $I = M^2$ for some maximal ideal M of R or $I = MM'$ for some maximal ideals M, M' of R .

(2) \Rightarrow (3) is straightforward.

(3) \Rightarrow (1) Suppose that (3) holds. Assume that there is an ideal J of R which satisfies $M^2 \subseteq I \subseteq M$. Then I is an M -primary ideal of R ; so I is a 2-absorbing ideal by [8, Theorem 3.1]. Hence $I = M$ or $I = M^2$ from our assumption (3). Thus there are no ideals properly between M and M^2 . From [12, (39.2) Theorem], R is a general ZPI-ring. \square

Theorem 7. Let R be a zero dimensional general ZPI-ring. Then $TZ\Gamma(R) = \emptyset$ if and only if either R is an integral domain or $0 = P^2$ where P is a prime ideal of R or $0 = PQ$ where P and Q are prime ideals of R .

Proof. If R is an integral domain or $0 = P^2$ where P is a prime ideal of R or $0 = PQ$ where P and Q are prime ideals of R , then it is easy to verify that there is no triple zero elements of R ; so $TZ\Gamma(R) = \emptyset$. Conversely, suppose that $TZ\Gamma(R) = \emptyset$. Then 0 is a 2-absorbing ideal of R by Theorem 1. From Lemma 6, either 0 is prime, $0 = P^2$ for some prime ideal P or $0 = PQ$ for some prime ideals P, Q of R , so we are done. \square

We recall that a special primary is an indecomposable general ZPI-ring which is a local ring with maximal ideal M such that each proper ideal of R is a power of M .

Lemma 8. [12] *An indecomposable general ZPI-ring with identity is either a Dedekind domain or a special primary ring.*

Theorem 9. *Let R be a general ZPI-ring and $0 = P^3$ where P is a prime ideal of R such that $P^2 \neq 0$. Then $TZ\Gamma(R)$ is a complete graph on $|P| - |P^2|$ vertices; i.e. $TZ\Gamma(R) \cong K_{|P| - |P^2|}$*

Proof. Suppose that $0 = P^3$ where P is a prime ideal of R . It is well-known that a ring R is indecomposable if and only if 1 is the only non-zero idempotent element of R . Let $0 \neq a \in R$ and $a^2 = a$. Hence $a - a^2 = a(1 - a) = 0 \in P$ implies $a \in P$ or $(1 - a) \in P$. If $a \in P$, then we get $0 = a^3 = a^2 = a$, a contradiction. Thus $(1 - a) \in P$. It follows $0 = (1 - a)^3 = 1 - 2a^2 + 2a - a^3 = 1 - a$, and so $a = 1$. Therefore, R is an indecomposable ring which is clearly not a domain as $0 = P^3$ and P is nonzero. Hence, we conclude from Lemma 8 that R is a special primary ring. Let M be the unique maximal ideal of R . Since every ideal, in particular, the zero ideal is a power of M , we have $M \subseteq \sqrt{0}$. Since $0 = P^3$, clearly we have $P = \sqrt{0} = M$.

Now, we show that a is a vertex of $TZ\Gamma(R)$ if and only if $a \in P \setminus P^2$. Let a be a vertex of $TZ\Gamma(R)$. Then, there exist $b, c \in R \setminus \{0\}$ such that $abc = 0$, $ab \neq 0$, $ac \neq 0$, $bc \neq 0$. If $a \notin P$, then a is unit and $bc = 0$ which is a contradiction. Thus $TZ(R) \subseteq P$. If $a \in P^2$, then since $b \in TZ(R) \subseteq P$, we conclude $ab \in P^3 = 0$, a contradiction. Therefore, $a \in P \setminus P^2$. Conversely, if $a \in P \setminus P^2$, then the claim follows from $a^3 = 0$ and $a^2 \in P^2 \neq 0$. Suppose a and b are any two distinct vertices. Since $a^2b = ab^2 = 0$ and ab, a^2, b^2 are nonzero, a and b are adjacent. Thus, $TZ\Gamma(R)$ is a complete graph on $|P| - |P^2|$ vertices. \square

Theorem 10. *Let $0 = P^2Q$ where P and Q are prime ideals of a general ZPI-ring R . Then $TZ\Gamma(R)$ is a connected graph with diameter 2 and girth 3.*

Proof. Suppose that $0 = P^2Q$. Let a be a vertex of $TZ\Gamma(R)$. We show that $a \in Q \setminus P$ or $a \in P \setminus (P^2 \cup Q)$. Since $a \in TZ(R)$, there exist $b, c \in R \setminus P^2Q$ such that $abc \in P^2Q$ and $ab, bc, ac \notin P^2Q$. Hence, we have either $a \in P$ or $b \in P$ or $c \in P$, and $a \in Q$ or $b \in Q$ or $c \in Q$.

Case I. Let $a \in P \cap Q$. If $a \in P^2$, then $a \in P^2 \cap Q = P^2Q = 0$ as P^2 and Q are coprime, a contradiction. So, assume that $a \in (P \setminus P^2) \cap Q$. If $b \in P$ or $c \in P$, then

$ab = 0$ or $ac = 0$, a contradiction. If $b \in Q \setminus P$ and $c \in Q \setminus P$, then we get $abc \notin P^2Q$ which is again a contradiction. Thus, $TZ(R) \subseteq (P \setminus Q) \cup (Q \setminus P)$.

Case II. Let $a \in P \setminus Q$. Suppose that $a \in P^2$. If $b \in Q \setminus P$ or $c \in Q \setminus P$, then we have either $ab = 0$ or $ac = 0$, a contradiction. If $b, c \in P \setminus Q$, then $abc \notin Q$, and so $abc \notin P^2Q$, a contradiction.

Therefore, we conclude that $a \in P \setminus (P^2 \cup Q)$ or $a \in Q \setminus P$.

Observe that all pairs are adjacent except for the elements of $Q \setminus P$. In fact, if an element $x \in TZ(R)$ satisfies $a_1b_1x = 0$, where $a_1, a_2 \in Q \setminus P$, we conclude that $x \in P^2$, a contradiction. Thus $TZ\Gamma(R)$ is a connected graph with $\text{diam}(TZ\Gamma(R)) = 2$ and $\text{gr}(TZ\Gamma(R)) = 3$. \square

In the next theorem, we give a necessary and sufficient conditions for $TZ\Gamma(R)$ to be triangle free.

Theorem 11. *Let R be a zero dimensional general ZPI-ring. $TZ\Gamma(R)$ is triangle free if and only if one of the following statements is hold:*

- (1) R is an integral domain.
- (2) $0 = PQ$ for some distinct prime ideals P and Q of R .
- (3) $0 = P^2$ for some prime ideal P of R .
- (4) $0 = P^3$ for some prime ideal P of R such that $|P| = 4$ and $|P^2| = 2$.

Proof. (\Rightarrow). We investigate the following cases separately.

Case I. Suppose that 0 is divisible by at least three prime ideals of R , say P, Q and T . Then $p \sim q \sim t$ where $p \in P, q \in Q, t \in T$ forms a triangle.

Case II. If 0 is divisible by P^2 and Q , where P and Q are distinct prime ideals of R , then we obtain the triangle $p \sim q \sim kp$, where $p \in P, q \in Q$ and $1 \neq k \in R \setminus Q$.

Case III. Suppose that $0 = P^n$, where P is prime and $n \geq 3$. If $n = 3$, then this graph is complete by Theorem 9. If $0 = P^n$ ($n \geq 4$), then $p \sim p^2 \sim kp$, where $p \in P$ and $1 \neq k \in R \setminus P$ forms a triangle.

(\Leftarrow). If (1), (2) or (3) holds, then $TZ\Gamma(R) = \emptyset$ by Theorem 7. If (4) holds, then there are the only two vertices connected by an edge by Theorem 9; so $TZ\Gamma(R) \cong K_2$. \square

So we conclude the following result.

Corollary 12. *The diameter of the triple zero graph of a zero dimensional general ZPI-ring R is an element of $\{0, 1, 2\}$ and the girth of the triple zero graph of R is 3 or undefined.*

In the following result, we characterize the triple zero graph of \mathbb{Z}_n and calculate $|TZ\Gamma(\mathbb{Z}_n)|$ cardinality of the vertex set for some particular cases.

Theorem 13. *Let $R = \mathbb{Z}_n$ where n is a positive integer. Then the following statements hold:*

- (1) *If $n = p$ or $n = p^2$ or $n = pq$, then $TZ\Gamma(\mathbb{Z}_n) = \emptyset$.*

- (2) If $n = p^3$ where p is prime, then $TZ\Gamma(\mathbb{Z}_n)$ is a complete graph on $p^2 - p$ vertices.
- (3) If $n = p^2q$ where p and q are distinct prime integers, then $TZ\Gamma(\mathbb{Z}_n)$ is a connected graph with diameter 2 and girth 3.

Proof. (1) is clear by Theorem 7.

(2) The vertices of $TZ\Gamma(\mathbb{Z}_n)$ are kp , where $k \in \mathbb{Z}_{p^2}^* = \{k \in \mathbb{Z} : (k, p^2) = 1, k < p^2\}$. So the number of vertices can be calculated by Euler's function $\phi(p^2) = p(p - 1)$. Since $(kp)(mp)(tp) = 0$ for all $k, m, t \in \mathbb{Z}_{p^2}^*$ and neither $(kp)(mp) = 0$ nor $(kp)(tp) = 0$ nor $(mp)(tp) = 0$, there is an edge between all vertices. Thus the graph is complete; so it is K_{p^2-p} .

(3) Suppose that $n = p^2q$. Then $TZ\Gamma(\mathbb{Z}_n)$ is a connected graph with diameter 2 and girth 3 by Theorem 10. Observe that the vertices of this graph are of the form kq where $k \in \mathbb{Z}_{p^2}^* = \{k \in \mathbb{Z} : (k, p^2) = 1, k < p^2\}$ and of the form sp where $s \in \Omega = \{s \in \mathbb{Z} : (s, p) = (s, q) = 1 \text{ and } s < pq\}$. So the number of vertices is $|\Omega| + \phi(p^2) = |\Omega| + p^2 - p$. Moreover, the number of edges can be calculated as $\binom{|\Omega|}{2} + (p^2 - p)|\Omega|$. □

Theorem 14. Let $n > 0$ and $R = \mathbb{Z}_n$. Then the following statements are equivalent:

- (1) $TZ\Gamma(\mathbb{Z}_n)$ is triangle free.
- (2) Either $n = p$, $n = p^2$, $n = pq$, or $n = 8$, where p and q are distinct prime integers.

Proof. We investigate the following cases separately.

Case I. Suppose that n is divisible by at least three primes, say p, q , and r . Then $p \sim q \sim (n/pq)$ forms a triangle.

Case II. If n is divisible by p^2 and q , where p and q are distinct prime integers, then we obtain the triangle $p \sim q \sim kp$, where $(k, q) = 1$ and $k < pq$.

Case III. Suppose that $n = p^n$, where p is prime and $n \geq 3$. If $n = 3$, then this graph is complete by Theorem 9. If $n = p^n$, where $n \geq 3$, except from $p = 2$, then $p \sim p^2 \sim kp$, where $(k, p) = 1, k < p^{n-3}$ forms a triangle. Thus, $n = p, n = p^2, n = pq$, or $n = 8$.

Conversely, if $n = p, n = p^2$ or $n = pq$, then $TZ\Gamma(\mathbb{Z}_n) = \emptyset$ by Theorem 7. If $n = 8$, then 2 and 6 are the only vertices connected by an edge; and so the claim is clear. □

So we conclude the following result which shows that $TZ\Gamma(\mathbb{Z}_n)$ is connected with diameter at most 2.

Corollary 15. The diameter of the triple zero graph of \mathbb{Z}_n is an element of $\{0, 1, 2\}$ and the girth of the triple zero graph of \mathbb{Z}_n is 3 or undefined.

Now we can summarize these results by the table below. Let p and q be distinct prime integers and $\Omega = \{s \in \mathbb{Z} : (s, p) = (s, q) = 1 \text{ and } s < pq\}$.

TABLE 1. $TZ\Gamma(\mathbb{Z}_n)$ Summary Table

n	Number of vertices	Number of edges	Diam	Girth	Remarks
p or p^2 or pq	0	0	0	∞	$TZ\Gamma(\mathbb{Z}_n) = \emptyset$
8	2	1	1	∞	$2 \sim 6$
$p^3 (p \geq 3)$	$p^2 - p$	$\binom{p^2 - p}{2}$	2	3	$K_{p^2 - p}$
$p^2 q$	$ \Omega + p^2 - p$	$\binom{ \Omega }{2} + (p^2 - p) \Omega $	2	3	Connected
All others			2	3	Connected

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